

# CENTERED COMPLEXITY ONE HAMILTONIAN TORUS ACTIONS

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ABSTRACT. We consider symplectic manifolds with Hamiltonian torus actions which are “almost but not quite completely integrable”: the dimension of the torus is one less than half the dimension of the manifold. We provide a complete set of invariants for such spaces when they are “centered” (see below) and the moment map is proper. In particular, this classifies the moment map preimages of all sufficiently small open sets, which is an important step towards global classification. As an application, we construct a full packing of each of the Grassmanians  $\text{Gr}^+(2, \mathbb{R}^5)$  and  $\text{Gr}^+(2, \mathbb{R}^6)$  by two equal symplectic balls.

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## 1. INTRODUCTION

Let a torus  $T \cong (S^1)^{\dim T}$  act effectively on a symplectic manifold  $(M, \omega)$  by symplectic transformations with a moment map  $\Phi: M \rightarrow \mathfrak{t}^*$ , that is,

$$(1.1) \quad \iota(\xi_M)\omega = -d\langle \Phi, \xi \rangle$$

for every  $\xi$  in the Lie algebra  $\mathfrak{t}$  of  $T$ , where  $\xi_M$  is the corresponding vector field on  $M$ . The dimension of the torus is at most half the dimension of the manifold. The difference  $k = \frac{1}{2} \dim M - \dim T$  is half the dimension of the symplectic quotient  $\Phi^{-1}(\alpha)/T$  at a regular value  $\alpha \in \Phi(M)$ . We call this number  $k$  the **complexity**<sup>1</sup>; see Definition 1.2.

The cases when  $M$  is compact and the complexity is zero, also known as *symplectic toric manifolds* or *Delzant spaces*, are classified by their moment images [De1]. The first examples of complexity one spaces are compact symplectic surfaces (with no action). By Moser [Mo], these are classified by their genus and total area. Compact symplectic four manifolds with Hamiltonian circle actions were classified by the first author [K2]; also see [AH, Au1, Au2]. In the algebraic category, complexity one actions (of possibly non-abelian groups) were recently classified by Timashëv [T1, T2]. Among other works on Lie group actions of complexity zero or one are [I, De2, W, GSj, Kn] in the symplectic category; [KKMS, OW, R, FK, BB, LV] in the algebraic category; [F, OR] in the smooth category.

This paper is the first in a series of papers in which we study complexity one spaces of arbitrary dimension. In this paper we study the basic building blocks: the preimages under the moment map of sufficiently small open subsets in  $\mathfrak{t}^*$ . We provide invariants which determine these spaces up to an equivariant symplectomorphism. Moreover, our techniques apply to all centered complexity one spaces (see below).

In later papers, we will give a complete global classification of complexity one spaces. This will provide a basis from which to address global questions about complexity one spaces, such as

1. What is the space of automorphisms?
2. When is there a compatible Kähler structure?
3. When are two complexity one spaces symplectomorphic? equivariantly diffeomorphic? diffeomorphic?

In this paper, because we wish to restrict to the preimages of open subsets of  $\mathfrak{t}^*$ , we do not insist that our manifolds be compact. Instead, we assume that the moment map is proper as a map to an open convex set  $U \subset \mathfrak{t}^*$ , that is, that the preimage of every compact subset of  $U$  is compact. The connectedness and convexity theorems still hold in this generality.

**Definition 1.2.** Let  $T$  be a torus. A **proper Hamiltonian  $T$ -manifold** is a connected symplectic manifold  $(M, \omega)$  together with an effective action of  $T$ , an open convex subset  $U \subseteq \mathfrak{t}^*$ , and a proper moment map  $\Phi: M \rightarrow U$ . Here,  $\mathfrak{t}$  is the Lie algebra of  $T$  and  $\mathfrak{t}^*$  the dual space. For brevity, in this paper we call  $(M, \omega, \Phi, U)$  a **complexity  $k$  space**, where  $k = \frac{1}{2} \dim M - \dim T$ . An **isomorphism** between two such spaces over the same set  $U$  is an equivariant symplectomorphism that respects the moment maps.

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<sup>1</sup> We changed our earlier term *deficiency* to *complexity* in order to be consistent with the algebraic geometers' terminology.

*Example 1.3.* A compact symplectic manifold with a torus action and a moment map is a proper Hamiltonian  $T$ -manifold over  $\mathfrak{t}^*$ .

*Example 1.4.* Let  $(M, \omega, \Phi, U)$  be a proper Hamiltonian  $T$ -manifold. For any open convex subset  $V \subseteq U$ , the preimage  $\Phi^{-1}(V)$  is a proper Hamiltonian  $T$ -manifold over  $V$ . The fact that it is connected follows from the facts that  $\Phi: \Phi^{-1}(V) \rightarrow V \cap \Phi(M)$  is proper and its image and fibers are connected (see Theorem 2.3) by easy point-set topology.

We will now describe invariants of a complexity one space. Over sufficiently small subsets of  $\mathfrak{t}^*$ , these will be enough to characterize the space.

The **Liouville measure** on a  $2n$  dimensional symplectic manifold  $(M, \omega)$  is given by integration of the volume form  $\omega^n/n!$  with respect to the symplectic orientation. In the presence of a Hamiltonian action, the **Duistermaat-Heckman measure** is the push-forward of Liouville measure by the moment map. It is equal to the **Duistermaat-Heckman function** times Lebesgue measure on  $\mathfrak{t}^*$ .

Assume  $M$  is connected. For any value  $\alpha \in \Phi(M)$ , if the symplectic quotient  $\Phi^{-1}(\alpha)/T$  is not a single point it is homeomorphic to a connected closed oriented surface (see Proposition 6.1). The genus of this surface does not depend on  $\alpha$  (see Corollary 9.9); we call it the **genus** of the complexity one space.

The **stabilizer** of a point  $x \in M$  is the closed subgroup  $H = \{\lambda \in T \mid \lambda \cdot x = x\}$ . The isotropy representation at  $x$  is the linear representation of  $H$  on the tangent space  $T_x M$ . Points in the same orbit have the same stabilizer and their isotropy representations are linearly symplectically isomorphic; this isomorphism class is the **isotropy representation** of the orbit.

An orbit is **exceptional** if every nearby orbit in the same moment fiber has a strictly smaller stabilizer. Since each moment fiber is compact, it contains finitely many exceptional orbits. The **isotropy data** at  $\alpha \in U$  is the unordered list of isotropy representations of the exceptional orbits in  $\Phi^{-1}(\alpha)$ .

With these definitions on hand, let us state our main theorem, which gives necessary and sufficient conditions for two complexity one spaces to be locally isomorphic.

**Theorem 1** (Local Uniqueness). *Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces. Assume that their Duistermaat-Heckman measures are the same and that their genus and isotropy data over  $\alpha \in U$  are the same. Then there exists a neighborhood of  $\alpha$  over which the spaces are isomorphic.*

Here is a simple proof of Theorem 1 in the case that the torus action on  $\Phi^{-1}(\alpha)$  is free:

The symplectic quotient  $\Phi^{-1}(\alpha)/T$  is a symplectic surface. Its symplectic area is the value of the Duistermaat-Heckman function at  $\alpha$ . Together with the genus, this determine the surface.

The moment fiber  $Z := \Phi^{-1}(\alpha)$  is a principal  $T$ -bundle over the symplectic quotient. Its Chern class is given by the slope of the Duistermaat-Heckman function at  $\alpha$  [DH].

The pullback to the moment fiber  $Z$  of the symplectic form on the symplectic quotient is the restriction  $i_Z^* \omega$  of  $\omega$  to  $Z$ . By the equivariant coisotropic embedding theorem (see [W1, lecture 5]). a neighborhood of the moment fiber  $Z$  is determined up to equivariant symplectomorphism by  $(Z, i_Z^* \omega)$ . Since the moment map is

proper, this neighborhood contains the preimage of a neighborhood of  $\alpha$ .

This argument straightforwardly extends to the case that  $\alpha$  is a regular value of the moment map. The main volume of this paper consists of carefully extending the argument to singular values of the moment map.

Additionally, we prove a variation of the theorem for “centered spaces” over larger subsets of  $\mathfrak{t}^*$ . Recall that the **orbit type strata** are the connected components of the sets of points with the same stabilizer.

**Definition 1.5.** A proper Hamiltonian  $T$ -manifold  $(M, \omega, \Phi, U)$  is **centered** about a point  $\alpha \in U$  if  $\alpha$  is contained in the closure of the moment image of every orbit type stratum in  $M$ .

**Theorem 2** (Centered Uniqueness). *Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces that are centered about  $\alpha \in U$ . Assume that their Duistermaat-Heckman measures are the same and that their genus and isotropy data over  $\alpha \in \mathfrak{t}^*$  are the same. Then the spaces are isomorphic.*

Finally, we present an application of our results to symplectic topology: we construct full packings of two Grassmanians. Holomorphic techniques in symplectic topology are useful in giving obstructions to embeddings, but have been less successful in constructing embeddings. Our construction, like several previous ones [T, K1], uses equivariant techniques to solve this non-equivariant problem. In a future paper [KT], we will extend the techniques that we develop here to address this question more deeply. Here, we content ourselves with a simple, but fairly representative, application:

**Theorem 3.** *Let  $M$  be the Grassmanian  $\mathrm{Gr}^+(2, \mathbb{R}^5)$  or  $\mathrm{Gr}^+(2, \mathbb{R}^6)$ . There exists an equivariant symplectic embedding of a disjoint union of two symplectic balls with linear actions and with equal radii into  $M$  such that the complement of the image has zero volume. A fortiori, each of these Grassmanians can be fully packed by two equal symplectic balls.*

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## 2. BACKGROUND

We now set our notation and review some background material.

Let a torus  $T$  act effectively on a symplectic manifold  $(M, \omega)$ . The symplectic slice at  $x \in M$  is the symplectic vector space

$$(T_x \mathcal{O})^\omega / (T_x \mathcal{O} \cap (T_x \mathcal{O})^\omega),$$

where  $\mathcal{O}$  is the  $T$ -orbit of  $x$  in  $M$ . Let  $H \subset T$  be the stabilizer of  $x$ . The isotropy representation of  $H$  on  $T_x M$  induces a representation on the symplectic slice, called the **slice representation**. The slice representation is isomorphic to the action of  $H$  on  $\mathbb{C}^n$  through an inclusion  $\rho = (\rho_1, \dots, \rho_n): H \rightarrow (S^1)^n$ . The isotropy representation is the direct sum of the slice representation and a trivial representation. The isotropy characters  $\rho_i$  are determined up to permutation. The differential of each  $\rho_i: H \rightarrow S^1$  is an element  $\eta_i$  of the dual space  $\mathfrak{h}^*$ . The  $\eta_i$  are called the **isotropy weights**.

We fix an inner product on the Lie algebra  $\mathfrak{t}$  of our torus  $T$ , once and for all. This determines a projection  $\mathfrak{t} \longrightarrow \mathfrak{h}$  and, dually, an inclusion  $\mathfrak{h}^* \hookrightarrow \mathfrak{t}^*$  for any subspace  $\mathfrak{h} \subset \mathfrak{t}$ . Throughout this paper, we will identify  $\mathfrak{h}^*$  with its image in  $\mathfrak{t}^*$ .

The Guillemin-Sternberg-Marle local normal form theorem classifies the neighborhoods of orbits in symplectic manifolds with Hamiltonian actions of compact groups. We state it for tori:

**Theorem 2.1** (Local normal form). *Let a closed subgroup  $H$  of a torus  $T$  act on  $\mathbb{C}^n$  by an inclusion  $\rho: H \longrightarrow (S^1)^n$  with weights  $\eta_1, \dots, \eta_n$  and moment map*

$$\Phi_H(z) = \frac{1}{2} \sum_{j=1}^n |z_j|^2 \eta_j.$$

1. *Equip  $T^*(T) \times \mathbb{C}^n$  with the standard symplectic form and the diagonal  $H$  action. Its symplectic quotient by  $H$  can be identified with the model*

$$Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0.$$

*Given  $\alpha \in \mathfrak{t}^*$ ,*

$$\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu$$

*is a moment map for the left  $T$  action. Here,  $T^*(T) = T \times \mathfrak{t}^*$  is the cotangent bundle of  $T$*

2. *Let the torus  $T$  act effectively on a symplectic manifold  $(M, \omega)$  with a moment map  $\Phi: M \longrightarrow \mathfrak{t}^*$ . Given a point  $x \in M$  with slice representation  $\rho$ , there exists a neighborhood of the orbit  $T \cdot x$  that is equivariantly symplectomorphic to a neighborhood of the orbit  $\{[t, 0, 0]\}$  in the model  $Y$  with  $\alpha = \Phi(x)$ .*

This is proved in [GS2] and in [M].

The local normal form theorem implies an important special case of Theorem 1:

**Proposition 2.2.** *Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be proper Hamiltonian  $T$ -manifolds. Consider a value  $\alpha \in U$  so that the moment fibers  $\Phi^{-1}(\alpha)$  and  $\Phi'^{-1}(\alpha)$  each consists of a single orbit. Suppose that these orbits have the same slice representation. Then there exists a neighborhood  $V$  of  $\alpha$  over which  $M$  and  $M'$  are isomorphic.*

*Proof.* Denote the orbits over  $\alpha$  in  $M$  and  $M'$  by  $\mathcal{O}$  and  $\mathcal{O}'$ . A neighborhood of  $\mathcal{O}$  in  $M$  and a neighborhood of  $\mathcal{O}'$  in  $M'$  are each isomorphic to a neighborhood of  $\{[t, 0, 0]\}$  in the same local model  $Y$ , by assumption and by the local normal form theorem. Hence a neighborhood  $W$  of  $\mathcal{O}$  is isomorphic to a neighborhood  $W'$  of  $\mathcal{O}'$ . Since the moment maps are proper, if  $V \subseteq U$  is a small enough neighborhood of  $\alpha$ , the preimages  $\Phi^{-1}(V)$  and  $\Phi'^{-1}(V)$  are contained in  $W$  and  $W'$ , and are hence isomorphic.  $\square$

We will also use the following global properties:

**Theorem 2.3.** *Every proper Hamiltonian  $T$ -manifold  $(M, \omega, \Phi, U)$  has the following properties.*

**Convexity:** *The moment image,  $\Phi(M)$ , is convex.*

**Connectedness:** *The moment fiber,  $\Phi^{-1}(\alpha)$ , is connected for all  $\alpha \in U$ .*

**Stability:** *As a map to  $\Phi(M)$ , the moment map is open.*

These properties, which are intimately related to each other, are due to Atiyah, Guillemin and Sternberg in the compact case. For Convexity and Connectedness, see [At]. For Convexity and Stability, see [GS1]. For proper moment maps to open convex sets and a brief history, see [LMTW].

### 3. ELIMINATING THE SYMPLECTIC FORM

Our next task is to free ourselves from the symplectic form. In this section we show that, instead of working with equivariant symplectomorphisms, it is enough to work with equivariant diffeomorphisms that respect the orientation and the moment map. These are much easier to work with, as one can apply techniques from differential topology.

**Definition 3.1.** Let  $M$  and  $M'$  be oriented manifolds with  $T$  actions and  $T$ -invariant maps  $\Phi: M \rightarrow \mathfrak{t}^*$  and  $\Phi': M' \rightarrow \mathfrak{t}^*$ . A  **$\Phi$ - $T$ -diffeomorphism** from  $(M, \Phi)$  to  $(M', \Phi')$  is an orientation preserving equivariant diffeomorphism  $\Psi: M \rightarrow M'$  that satisfies  $\Psi^*(\Phi') = \Phi$ .

In this section and the next one we will need to assume the following technical condition:

$$(3.2) \quad \begin{array}{l} \text{The restriction map } H^2(M/T, \mathbb{Z}) \rightarrow H^2(\Phi^{-1}(y)/T, \mathbb{Z}) \\ \text{is one-to-one for some regular value } y \text{ of } \Phi. \end{array}$$

In fact, this restriction map is always one-to-one. We prove this for the preimage of small open sets in this paper; see Corollary 9.8 and Lemma 5.6. In a later paper, we will prove it for all complexity one spaces.

**Proposition 3.3.** *Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces that satisfy Condition (3.2). Assume that they have the same Duistermaat-Heckman measure.<sup>2</sup> Then there exists an equivariant symplectomorphism from  $M$  to  $M'$  if and only if there exists a  $\Phi$ - $T$ -diffeomorphism from  $M$  to  $M'$ .*

The following proof of Proposition 3.3 relies on a couple of technical lemmas which we postpone until after the proof.

*Proof.* Let  $g: M \rightarrow M'$  be a  $\Phi$ - $T$ -diffeomorphism. By Lemma 3.5, there exists a basic one-form  $\beta$  on  $M$  (see Remark 3.4) such that  $d\beta = g^*\omega' - \omega$ . We now apply Moser's method:

Define  $\omega_t := (1-t)\omega + tg^*\omega'$  for all  $0 \leq t \leq 1$ . By Lemma 3.6 below, the  $\omega_t$  are nondegenerate. Let  $X_t$  be the vector field determined by  $i_{X_t}\omega_t = -\beta$ . The vector field  $X_t$  preserves the level sets of  $\Phi$ , because for every  $\xi \in \mathfrak{t}$ ,  $\langle d\Phi(X_t), \xi \rangle = -\omega_t(\xi_M, X_t) = -i_{\xi_M}\beta = 0$ . Since  $\Phi$  is proper, the time-dependent vector-field  $X_t$  integrates to a flow,  $F_t: M \rightarrow M$ . Let  $g_t = g \circ F_t$ . Then  $\Phi' \circ g_t = \Phi$ .

The vector field  $X_t$  is invariant because  $\omega_t$  and  $\beta$  are invariant. Consequently,  $g_t$  is equivariant, and hence is a  $\Phi$ - $T$ -diffeomorphism. Finally,

$$\begin{aligned} \frac{d}{dt}(F_t^*\omega_t) &= F_t^*(L_{X_t}\omega_t) + F_t^*\left(\frac{d}{dt}\omega_t\right) \\ &= F_t^*d\iota_{X_t}\omega_t + F_t^*(\omega_1 - \omega_0) \\ &= F_t^*(-d\beta + d\beta) = 0. \end{aligned}$$

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<sup>2</sup> In fact, we only need that the Duistermaat-Heckman functions agree at a point. Contrast with footnote 3.

Therefore,  $F_1^*\omega_1 = F_0^*\omega_0 = \omega_0$ , since  $F_0$  is the identity. Then,  $g_1^*\omega' = F_1^*(g^*\omega') = F_1^*(\omega_1) = \omega_0 = \omega$ . In other words,  $g_1$  is an (equivariant) symplectomorphism.  $\square$

Before proving the technical lemmas used in the above proof, let us recall the notion of basic forms:

*Remark 3.4.* Let a compact Lie group  $G$  act on a manifold  $M$ , and for  $\xi \in \mathfrak{g}$  let  $\xi_M$  be the generating vector-fields. A differential form  $\beta$  on  $M$  is **basic** if it is  $G$  invariant and horizontal, that is,  $\iota_{\xi_M}\beta = 0$  for all  $\xi \in \mathfrak{g}$ . The basic differential forms on  $M$  constitute a differential complex  $\Omega_{\text{basic}}^*(M)$  whose cohomology coincides with the Čech cohomology of the topological quotient,  $M/G$ . See [Kl]. To see this, repeat the standard Čech-de Rham spectral-sequence argument, as in [BT]. It still works because, by the local normal form for smooth actions of compact Lie groups, every orbit in  $M$  has a neighborhood on which the complex of basic forms is acyclic.

**Lemma 3.5.** *Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces. Assume that the Duistermaat-Heckman measures for  $M$  and  $M'$  are the same, and that the spaces satisfy Condition (3.2). Then for every  $\Phi$ - $T$ -diffeomorphism  $g: M \rightarrow M'$  there exists a basic one-form  $\beta$  on  $M$  such that  $d\beta = g^*\omega' - \omega$ .*

*Proof.* Let  $\Omega = g^*\omega' - \omega$ . Since  $\omega$  and  $g^*\omega'$  are closed invariant symplectic forms on  $M$  with the same moment map,  $\iota(\xi_M)\Omega = 0$  for all  $\xi \in \mathfrak{t}$ . Since  $\Omega$  is also invariant, it is basic.

By Condition (3.2) it suffices to show that the restriction of  $\Omega$  to the fiber  $\Phi^{-1}(\alpha)$  is exact for some regular value  $\alpha$  of  $\Phi$ . Since this restriction is the pull-back of a differential form  $\Omega_{\text{red}}$  on the orbifold  $M_{\text{red}} = \Phi^{-1}(\alpha)/T$ , it is enough to show that  $\Omega_{\text{red}}$  is exact. Since  $M_{\text{red}}$  is two dimensional, it is enough to show that the integral of  $\Omega_{\text{red}}$  over it is zero, i.e., that the integrals of  $\omega$  and  $g^*\omega'$  are equal. But this follows from the fact that the Duistermaat-Heckman measures for  $M$  and  $M'$  are the same, because the density functions for these measures are given by the symplectic volumes of the symplectic quotients; see [DH, §3].  $\square$

**Lemma 3.6.** *Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces. Let  $g: M \rightarrow M'$  be a  $\Phi$ - $T$ -diffeomorphism. Then the two-form  $\omega_t = (1-t)\omega + tg^*\omega'$  is nondegenerate for all  $0 \leq t \leq 1$ .*

*Proof.* First, let a compact abelian Lie group  $H$  act on  $\mathbb{C}^n$  as a codimension one subgroup of  $(S^1)^n$  with isotropy weights  $\eta_1, \dots, \eta_n \in \mathfrak{h}^*$ . The vector fields for this action are

$$\xi_M = \sqrt{-1} \sum_i \eta_i(\xi) \left( z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right), \quad \xi \in \mathfrak{h}.$$

Let  $\tilde{\omega}_0$  and  $\tilde{\omega}_1$  be invariant symplectic forms on  $\mathbb{C}^n$  with constant coefficients that have the same moment map,  $\Phi_H$ , and that induce the same orientation. We will show that  $\tilde{\omega}_t = (1-t)\tilde{\omega}_0 + t\tilde{\omega}_1$  is non-degenerate.

Because  $\tilde{\omega}_t$  is real valued, it can be written in the form

$$\begin{aligned} \tilde{\omega}_t &= \sqrt{-1} \sum_j A_j^t dz_j \wedge d\bar{z}_j \\ &+ \frac{1}{2} \sqrt{-1} \sum_{j \neq k} \left( B_{jk}^t dz_j \wedge dz_k - \overline{B_{jk}^t} d\bar{z}_j \wedge d\bar{z}_k \right) \\ &+ \sqrt{-1} \sum_{j \neq k} C_{jk}^t dz_j \wedge d\bar{z}_k, \end{aligned}$$

where  $A_j^t$  are real,  $B_{jk}^t$  and  $C_{jk}^t$  are complex,  $B_{jk}^t = -B_{kj}^t$ , and  $C_{jk}^t = \overline{C_{kj}^t}$ .

By the definition of moment map,  $\partial\Phi_H^\xi/\partial z_j$  is the coefficient of  $dz_j$  in  $-\iota(\xi_M)\tilde{\omega}_t$ . Hence

$$\partial\Phi_H^\xi/\partial z_j = \eta_j(\xi)A_j^t\bar{z}_j + \sum_{k \neq j} \eta_k(\xi) (B_{kj}^t z_k + C_{jk}^t \bar{z}_k).$$

Differentiating again,  $\partial^2\Phi_H^\xi/\partial z_j\partial\bar{z}_j = \eta_j(\xi)A_j^t$ . Because  $\tilde{\omega}_t$  is  $T$ -invariant,  $B_{jk}^t = 0$  unless  $\eta_j = -\eta_k$ . In this case,  $\partial^2\Phi_H^\xi/\partial z_j\partial z_k = \eta_k(\xi)B_{kj}^t$ . Finally,  $C_{jk}^t = 0$  unless  $\eta_j = \eta_k$ ; in this case,  $\partial^2\Phi_H^\xi/\partial z_j\partial\bar{z}_k = \eta_k(\xi)C_{jk}^t$ . Therefore,  $\eta_j(\xi)A_j^t$ ,  $\eta_k(\xi)B_{kj}^t$ , and  $\eta_k(\xi)C_{jk}^t$  are determined by  $\Phi_H^\xi$ .

By what we have shown, if  $\eta_j \neq 0$ , the coefficients  $A_j^t$ ,  $B_{jk}^t$ , and  $C_{jk}^t$  are determined by  $\Phi_H$ . Thus, if no weight is zero,  $\tilde{\omega}_t$  is independent of  $t$ , hence it is non-degenerate. Since the action is effective and the dimension of  $\mathfrak{h}^*$  is  $n-1$ , the only other possibility is that exactly one of the weights – let's say the first one – is zero, and the others form a basis of  $\mathfrak{h}^*$ . In this case,  $B_{ij}^t = C_{ij}^t = 0$  for all  $i$  and  $j$ , and so the top power of  $\tilde{\omega}_t$  is  $\prod_{j=1}^n A_j^t$  times the standard volume form. Since  $\tilde{\omega}_0$  and  $\tilde{\omega}_1$  induce the same orientation, and since  $A_j^t$  is determined by  $\Phi_H$  and hence independent of  $t$  for  $j \neq 1$ , the signs of  $A_1^0$  and  $A_1^1$  are the same. Therefore,  $A_j^t = (1-t)A_j^0 + tA_j^1$  is never zero, and  $\tilde{\omega}_t$  is non-degenerate.

Now let  $x \in M$  be any point with stabilizer  $H \subset T$ . By the local normal form theorem, a neighborhood of the orbit  $T \cdot x$  in  $M$  with the symplectic form  $\omega_0$  is equivariantly symplectomorphic to a neighborhood of the orbit  $\{[t, 0, 0]\}$  in the model  $T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ . The tangent space at  $x$  splits as

$$T_x M = \mathfrak{t}/\mathfrak{h} \oplus \mathfrak{h}^0 \oplus \mathbb{C}^n$$

where  $\mathfrak{t}/\mathfrak{h}$  is the tangent to the orbit. By the definition of the moment map,  $\omega_t|_x$  is given by a block matrix of the form

$$\begin{pmatrix} 0 & I & 0 \\ -I & * & * \\ 0 & * & \tilde{\omega}_t \end{pmatrix}$$

where  $I$  is the natural pairing between the vector space  $\mathfrak{t}/\mathfrak{h}$  and its dual,  $\mathfrak{h}^0$ , and where  $\tilde{\omega}_0$  and  $\tilde{\omega}_1$  are linear symplectic forms on  $\mathbb{C}^n$  with the same moment map and the same orientation. By the above argument,  $\tilde{\omega}_t$  is nondegenerate. Consequently,  $\omega_t|_x$  is nondegenerate.  $\square$

#### 4. PASSING TO THE QUOTIENT

In this section we show that, as long as two complexity one spaces have the same Duistermaat-Heckman measure, we can reduce the problem of finding a  $\Phi$ - $T$ -diffeomorphism between them to the easier problem of finding a  $\Phi$ -diffeomorphism between their quotients. Some techniques in this section are adapted from Haefliger and Salem [HS].

Let a compact torus  $T$  act on a manifold  $N$ . The quotient  $N/T$  can be given the quotient topology and a natural differential structure, consisting of the sheaf of real-valued functions whose pullbacks to  $N$  are smooth. We say that a map  $h: N/T \rightarrow N'/T$  is smooth if it pulls back smooth functions to smooth functions; it is a diffeomorphism if it is smooth and has a smooth inverse. See [Sch2]. If  $N$  and  $N'$  are oriented, the choice of an orientation on  $T$  determines orientations on



the smooth part of  $N/T$  and  $N'/T$ . Whether or not a diffeomorphism  $f: N/T \rightarrow N'/T$  preserves orientation is independent of this choice.

While this notion of diffeomorphism is natural, we will also need another less natural but stronger notion.

**Definition 4.1.** Let  $M$  and  $M'$  be oriented manifolds with  $T$  actions and  $T$ -invariant maps  $\Phi: M \rightarrow \mathfrak{t}^*$  and  $\Phi': M' \rightarrow \mathfrak{t}^*$ . A  **$\Phi$ -diffeomorphism** from  $M/T$  to  $M'/T$  is an orientation preserving diffeomorphism  $\Psi: M/T \rightarrow M'/T$  such that

1.  $\Psi$  preserves the moment map, i.e.,  $\Psi^*\Phi' = \Phi$ .
2. Each of  $\Psi$  and  $\Psi^{-1}$  lifts to a  $\Phi$ - $T$ -diffeomorphism in a neighborhood of each exceptional orbit.

**Proposition 4.2.** Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces. Assume that Condition (3.2) is satisfied. If both spaces have the same Duistermaat-Heckman measure,<sup>3</sup> every  $\Phi$ -diffeomorphism from  $M/T$  to  $M'/T$  lifts to  $\Phi$ - $T$ -diffeomorphism from  $M$  to  $M'$ .

The first step in proving this proposition is to show that on the non-exceptional orbits every  $\Phi$ -diffeomorphism lifts locally to a  $\Phi$ - $T$ -diffeomorphism. We do this in the next three lemmas.

**Lemma 4.3.** Every local model for a non-exceptional orbit in a complexity one space has the form

$$(4.4) \quad Y = T \times_H \mathbb{C}^h \times \mathbb{C} \times \mathfrak{h}^0,$$

where  $H \subseteq T$  is a closed  $h$  dimensional subgroup which acts on  $\mathbb{C}^h$  through an isomorphism with  $(S^1)^h$ .

*Proof.* Let  $Y := T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$  be the local model for a non-exceptional orbit in  $\Phi^{-1}(\alpha)$ , with  $h = \dim H$ . Inside the moment fiber  $\Phi_Y^{-1}(\alpha)$ , the set of points with stabilizer  $H$  is

$$(4.5) \quad T \times_H (\mathbb{C}^{h+1})^H \times \{0\},$$

where  $(\mathbb{C}^{h+1})^H$  is the subspace fixed by  $H$ . By the definition of exceptional orbit, this subspace is not trivial. Therefore, the local model becomes (4.4), where the group  $H$  acts trivially on  $\mathbb{C}$  and acts on  $\mathbb{C}^h$  through an inclusion into  $(S^1)^h$ . By a dimension count, this inclusion must be an isomorphism.  $\square$

The following lemma tells us that neighborhoods of nonexceptional orbits can be read off from the moment image.

**Lemma 4.6.** Let  $(M, \omega, \Phi, U)$  be a complexity one space. Assume that the preimage  $\Phi^{-1}(\alpha)$  of  $\alpha \in U$  contains a non-exceptional orbit.

There exists a closed connected subgroup  $H \subseteq T$  with Lie algebra  $\mathfrak{h}$  and a basis  $\{\eta_j\}$  for the weight lattice in  $\mathfrak{h}^*$  so that

1. The group  $H$  is the stabilizer and the  $\eta_j$  are the non-zero isotropy weights of every non-exceptional orbit in  $\Phi^{-1}(\alpha)$ .
2. In a neighborhood of  $\alpha$ , the image  $\Phi(M)$  coincides with the Delzant cone  $\alpha + \mathfrak{h}^0 + \sum_j \mathbb{R}_+ \eta_j$ .

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<sup>3</sup> In fact, we only need their Duistermaat-Heckman functions to have the same slope; they may differ by a constant. Contrast with footnote 2.

*Proof.* Consider the slice representation at any non-exceptional orbit. By Lemma 4.3 above, the stabilizer  $H$  is connected. Since the action is effective, the isotropy weights  $\eta_j$  generate the weight lattice. The stabilizer and these weights are determined by the image of the moment map for a local model; this image is the Delzant cone  $\mathfrak{h}^0 + \sum_j \mathbb{R}_+ \eta_j$ . Finally, by the stability of the moment map, the image of the moment map is the same for every local model in  $\Phi^{-1}(\alpha)$ .  $\square$

**Corollary 4.7.** *Over the interior of the moment image, the nonexceptional orbits are precisely the free orbits.*

**Lemma 4.8.** *Let  $Y$  be a local model for a non-exceptional orbit with a moment map  $\Phi_Y: Y \rightarrow \mathfrak{t}^*$ . Let  $W$  and  $W'$  be invariant open subsets of  $Y$ . Let  $g: W/T \rightarrow W'/T$  be a diffeomorphism which preserves the moment map. Then  $g$  lifts to an equivariant diffeomorphism from  $W$  to  $W'$ .*

*Proof.* Assume  $W = W' = Y$ ; the general case is similar. Since, by Lemma 4.3,  $Y = T \times_H \mathbb{C}^h \times \mathbb{C} \times \mathfrak{h}^0$ , we can identify  $Y/T$  with  $\mathfrak{h}^0 \times (\mathbb{C}^h/H) \times \mathbb{C}$ . Since  $g$  preserves the moment map, it necessarily has the form

$$g(\nu, [z], \zeta) = (\nu, [z], \psi(\nu, [z], \zeta)),$$

for some  $\psi: \mathfrak{h}^0 \times (\mathbb{C}^h/H) \times \mathbb{C} \rightarrow \mathbb{C}$ . Similarly, its inverse sends  $(\nu, [z], \zeta)$  to  $(\nu, [z], \gamma(\nu, [z], \zeta))$ , where  $\gamma: \mathfrak{h}^0 \times (\mathbb{C}^h/H) \times \mathbb{C} \rightarrow \mathbb{C}$ . Since both  $g$  and its inverse are smooth,  $\zeta$  and  $\gamma$  must themselves be smooth.

We define  $\tilde{g}: Y \rightarrow Y$  by  $\tilde{g}([t, z, \zeta, \nu]) = [t, z, \psi(\nu, [z], \zeta), \nu]$ . Then  $\tilde{g}$  is a smooth equivariant lift of  $g$ , and it has a smooth inverse given by  $[t, z, \zeta, \nu] \mapsto [t, z, \gamma(\nu, [z], \zeta), \nu]$ .  $\square$

We deduce that a  $\Phi$ -diffeomorphism lifts to a  $\Phi$ - $T$ -diffeomorphism locally; we still need to show that, in the proper circumstances, a  $\Phi$ -diffeomorphism that lifts locally also lifts globally. We do this in the lemma below; the basic idea is that the Duistermaat-Heckman measure determines the “fibration”  $M \rightarrow M/T$ .

**Lemma 4.9.** *Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces. Assume that Condition (3.2) is satisfied. If both spaces have the same Duistermaat-Heckman measure, every homeomorphism from  $M/T$  to  $M'/T$  that locally lifts to a  $\Phi$ - $T$ -diffeomorphism also lifts globally to a  $\Phi$ - $T$ -diffeomorphism from  $M$  to  $M'$ .*

*Proof.* Let  $T, \mathfrak{t}$ , and  $\ell$  denote the sheaves of smooth functions from  $M/T$  to  $T, \mathfrak{t}$ , and  $\ell$ , respectively. Here,  $\ell$  denotes the lattice in  $\mathfrak{t}$ . Let  $\underline{\mathfrak{t}}$  denote the sheaf of locally constant function to  $\mathfrak{t}$ .

Fix a homeomorphism  $\Psi: M/T \rightarrow M'/T$  that lifts locally. Choose a cover  $\mathcal{U}$  of  $M$  by open invariant sets, and on each  $U_i \in \mathcal{U}$  a  $\Phi$ - $T$ -diffeomorphism  $\Psi_i: U_i \rightarrow M'$  that is a lift of  $\Psi$ . By Lemma 4.11 below, there exist smooth invariant functions  $g_{ij}: U_i \cap U_j \rightarrow T$  such that  $g_{ij} \cdot \Psi_j = \Psi_i$  for all  $i$  and  $j$ . These functions form a Čech cocycle  $g \in \check{C}^1(\mathcal{U}, T)$ . The map  $\Psi$  will lift to a global  $\Phi$ - $T$ -diffeomorphism exactly if the corresponding cohomology class  $[g] \in \check{H}^1(M/T, T)$  is trivial.

The short exact sequence  $0 \rightarrow \ell \rightarrow \mathfrak{t} \rightarrow T \rightarrow 0$  induces a long exact sequence in cohomology. Since there exists a smooth partition of unity on  $M/T$ , the cohomology  $\check{H}^i(M/T, \mathfrak{t})$  vanishes for all  $i > 0$ . Therefore,  $\check{H}^1(M/T, T) = \check{H}^2(M/T, \ell)$ . Condition (3.2) implies that the restriction map  $\check{H}^2(M/T, \ell) \rightarrow \check{H}^2(\Sigma, \ell)$  is one-to-one, where  $\Sigma = \Phi^{-1}(\alpha)/T$  is a regular symplectic quotient.

Therefore, it is enough to show that the image of  $[g]$  in  $\check{H}^2(\Sigma, \ell)$  is zero. Since  $\check{H}^2(\Sigma, \ell)$  is torsion free, it is enough to show that the image of  $[g]$  in  $\check{H}^2(\Sigma, \mathfrak{t})$  vanishes. The Čech-de Rham isomorphism for basic forms on  $\Phi^{-1}(\alpha)$  (see Remark 3.4) takes this image to the cohomology class of the basic differential two-form whose restriction to each open set  $U_i \cap \Phi^{-1}(\alpha)$  is

$$(4.10) \quad \pm \sum_j d\lambda_j g_{ij}^{-1} dg_{ij},$$

(the sign depending on conventions), where  $\{\lambda_i\}$  is a partition of unity subordinate to  $\mathcal{U} \cap \Phi^{-1}(\alpha)$ . We claim that this is exact as a basic form.

Let  $\Theta'$  be a connection one-form on  $\Phi'^{-1}(\alpha) \subset M'$ , that is, a  $T$ -invariant  $\mathfrak{t}$ -valued one-form such that  $\Theta'(\xi_{M'}) \equiv \xi$  for all  $\xi \in \mathfrak{t}$ . Then  $\Theta = \sum \lambda_i \Psi_i^* \Theta'$  is a connection one-form on  $\Phi^{-1}(\alpha) \subset M$ . The curvature forms  $d\Theta$  and  $d\Theta'$  are basic. Their integrals over the symplectic quotients are equal to the slopes of the Duistermaat-Heckman function of  $M$  and of  $M'$  at  $\alpha$  [DH]. Since these slopes are the same, and since  $\Phi^{-1}(\alpha)/T$  is a two dimensional orbifold, the difference between  $d\Theta$  and  $\Psi^* d\Theta'$  is exact as a basic form. A simple computation shows that this difference is equal to (4.10).  $\square$

In the above proof we used the following theorem of Haefliger and Salem, based on a lemma of Schwarz.

**Theorem 4.11** ([HS]). *Let a torus  $T$  act on a manifold  $M$ . Let  $h: M \longrightarrow M$  be an equivariant diffeomorphism that sends each orbit to itself. Then there exists a smooth invariant function  $f: M \longrightarrow T$  such that  $h(m) = f(m) \cdot m$  for all  $m \in M$ .*

We are finally ready to prove our main proposition.

*Proof of Proposition 4.2.* Let  $\mathcal{O}$  be a non-exceptional orbit in  $M$ . By Definition 4.1, any  $\Phi$ -diffeomorphism sends it to a non-exceptional orbit  $\mathcal{O}'$  in  $M'$ . By Lemma 4.6, the local models for  $\mathcal{O}$  and  $\mathcal{O}'$  are the same  $Y$ . By Lemma 4.8 and the local normal form theorem, the map lifts to a  $\Phi$ - $T$ -diffeomorphism from a neighborhood of  $\mathcal{O}$  to a neighborhood of  $\mathcal{O}'$ . By Lemma 4.9, it lifts globally.  $\square$

## 5. SYMPLECTIC REPRESENTATIONS

So far, we have shown that two complexity one spaces with the same Duistermaat-Heckman measure are isomorphic if their quotients are  $\Phi$ -diffeomorphic as long as the spaces satisfy Condition (3.2). The rest of the paper is dedicated to proving that, over small subsets of  $\mathfrak{t}^*$ , the quotients are indeed  $\Phi$ -diffeomorphic if the spaces have the same genus and isotropy data, and that Condition (3.2) is always satisfied.

In preparation for this, in this section we analyze how the weights of a symplectic representation can be read from its moment map. The key ingredient, which we use repeatedly, is simply the formula for the moment map: let a compact abelian group  $H$  act effectively on  $\mathbb{C}^n$  as a subgroup of  $(S^1)^n$  with weights  $\eta_1, \dots, \eta_n$ . Then

$$(5.1) \quad \Phi_H(z) = \frac{1}{2} \sum_{j=1}^n |z_j|^2 \eta_j$$

is a moment map.

**Lemma 5.2.** *Let a compact abelian group  $H$  act effectively on  $\mathbb{C}^n$  with weights  $\eta_1, \dots, \eta_n$ . The moment map  $\Phi_H$  is onto if and only if there exist  $\xi_j > 0$  so that  $\sum \xi_j \eta_j = 0$ .*

*Proof.* Suppose that the moment map  $\Phi_H$  is onto. Then every element of  $\mathfrak{h}^*$  is in the non-negative span of the  $\{\eta_j\}$ . In particular, there exist  $a_j \geq 0$  such that  $\sum a_j \eta_j = \sum_j -\eta_j$ , that is,  $\sum (1 + a_j) \eta_j = 0$ . Let  $\xi_j = 1 + a_j$ .

Conversely, suppose that there exist positive  $\xi_j$ 's so that  $\sum \xi_j \eta_j = 0$ . Let  $\alpha \in \mathfrak{h}^*$  be any element. Because the action is effective, its weights,  $\eta_j$ , span  $\mathfrak{h}^*$ , so there exist  $a_j, j = 1, \dots, n$ , such that  $\alpha = \sum_j a_j \eta_j$ . Because  $\sum_j \xi_j \eta_j = 0$ , we also have  $\alpha = \sum_j (a_j + t \xi_j) \eta_j$  for any  $t \in \mathbb{R}$ . Because  $\xi_j > 0$  for all  $j$ , if we take  $t$  large enough we get that  $\alpha$  is in the positive span of the  $\eta_j$ .  $\square$

**Lemma 5.3.** *Let a compact abelian group  $H$  act effectively on  $\mathbb{C}^n$  with moment map  $\Phi_H$ . The moment map  $\Phi_H$  is not proper if and only if there exist  $\xi_j \geq 0$ , not all zero, such that  $\sum \xi_j \eta_j = 0$ .*

*Proof.* Suppose that  $\sum \xi_j \eta_j = 0$  for some  $\xi_j \geq 0$ , not all zero. Since the moment fiber  $\Phi_H^{-1}(0)$  contains the line  $(t(\xi_1)^{\frac{1}{2}}, \dots, t(\xi_k)^{\frac{1}{2}})$ ,  $t \in \mathbb{R}$ , the map is not proper.

Conversely, suppose that  $\sum \xi_i \eta_i \neq 0$  whenever  $\xi_i \geq 0$  are not all zero. Then  $m = \min\{|\Phi_H(z)|\}_{|z|^2=1}$  is positive. Since  $\Phi_H$  is quadratic,  $|\Phi_H(z)| \geq m|z|^2$  for all  $z$ , which implies that  $\Phi_H$  is proper.  $\square$

This analysis already distinguishes the two possibilities for non-empty moment fibers.

**Lemma 5.4.** *Let  $(M, \omega, \Phi, U)$  be a proper Hamiltonian  $T$ -manifold. For any  $\alpha \in U$ , if the moment fiber  $\Phi^{-1}(\alpha)$  is not empty, it consists of either*

1. *a single orbit, which has a local model with a proper moment map, or*
2. *infinitely many orbits, each of which has a local model with a non-proper moment map.*

*If  $\alpha \in \text{interior}(\Phi(M))$ , the second case occurs.*

*Proof.* Given  $x \in M$ , let  $\rho: H \rightarrow (S^1)^n$  be the slice representation. Consider the local model  $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$  with moment map

$$(5.5) \quad \Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu,$$

where  $\Phi_H$  is the moment map for  $\rho$  and where  $\alpha = \Phi(x)$ .

By Lemma 5.3 and equations (5.1) and (5.5), the moment map  $\Phi_Y$  is proper if and only if the moment fiber  $\Phi_Y^{-1}(\alpha)$  consists of a single orbit, and otherwise  $\Phi_Y^{-1}(\alpha)$  contains infinitely many orbits near  $\{[t, 0, 0]\}$ . The lemma now follows from the local normal form theorem and the connectedness of moment fibers.  $\square$

We will also use the following corollary of Lemma 5.4.

**Lemma 5.6.** *Let  $(M, \omega, \Phi, U)$  be a proper Hamiltonian  $T$ -manifold, and let  $\alpha$  be a point in  $U$  whose moment fiber  $\Phi^{-1}(\alpha)$  contains exactly one orbit. Then every neighborhood of  $\alpha$  which is contained in  $U$  contains a smaller neighborhood  $V$  whose preimage,  $\Phi^{-1}(V)$ , is contractible.*

*Proof.* Let  $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$  be the corresponding model. By Lemma 5.4, the moment map  $\Phi_Y$  is proper. By Proposition 2.2, the preimage in  $M$  and in  $Y$  of a sufficiently small neighborhood  $V$  of  $\alpha$  are isomorphic. Thus, we may work purely inside  $Y$ . Choose a neighborhood of  $\alpha$  of the form  $V = V_1 \times V_2$ , where  $V_1 \subset \mathfrak{h}^*$  and  $V_2 \subset \mathfrak{h}^0$  are convex, and where we identify  $\mathfrak{t}^* = \mathfrak{h}^* \times \mathfrak{h}^0$ . Because  $\Phi_H$  is homogeneous,  $\Phi_Y^{-1}(V)/T = (\Phi_H^{-1}(V_1)/T) \times V_2$  is contractible.  $\square$

We have already proved the local uniqueness theorem (Theorem 1) in case 1 of Lemma 5.4. This is Proposition 2.2. Therefore, for the rest of the proof of the theorem we may focus on case 2 of Lemma 5.4.

So far, we have been allowing actions of any complexity, but we now restrict to complexity one to define a useful polynomial:

**Lemma 5.7.** *Let an  $h$ -dimensional compact abelian Lie group  $H$  act on  $\mathbb{C}^{h+1}$  as a subgroup of  $(S^1)^{h+1}$  with a moment map that is not proper. Then there exists a unique polynomial*

$$(5.8) \quad P(z) = \prod_{j=0}^h z_j^{\xi_j},$$

with  $\xi_j \geq 0$  for all  $j$ , such that the following sequence is exact:

$$(5.9) \quad 1 \longrightarrow H \xrightarrow{\rho} (S^1)^{h+1} \xrightarrow{P} S^1 \longrightarrow 1.$$

Moreover,  $\xi_j > 0$  for all  $j$  exactly if the moment map is onto.

*Proof of Lemma 5.7.* Because the quotient  $(S^1)^{h+1}/H$  is a one dimensional compact connected Lie group, there exists a homomorphism  $P: (S^1)^{h+1} \longrightarrow S^1$  such that the sequence (5.9) is exact. Such a homomorphism must be of the form

$$(5.10) \quad P(\lambda) = \prod_j \lambda_j^{\xi_j}$$

for some integers  $\xi_0, \dots, \xi_h$ . Let  $\eta_j \in \mathfrak{h}^*$  denote the weights for the  $H$ -action on  $\mathbb{C}^{h+1}$ . Differentiating the identity  $P \circ \rho = 1$  from (5.9), we get

$$(5.11) \quad \sum_j \xi_j \eta_j = 0.$$

By Lemma 5.3, because the moment map  $\Phi_H$  is not proper, there exist non-negative numbers  $\xi'_0, \dots, \xi'_h$ , not all zero, such that  $\sum \xi'_j \eta_j = 0$ . Since  $\sum \xi_j \eta_j = 0$ , by a dimension count the vector  $(\xi_j)$  must be a multiple of the vector  $(\xi'_j)$ . Therefore, after possibly replacing the vector  $(\xi_j)$  by the vector  $(-\xi_j)$ , all the  $\xi_j$ 's are non-negative. By Lemma 5.2 and a similar dimension count, the  $\xi_j$  are strictly positive exactly if the moment map is onto.  $\square$

**Definition 5.12.** We call  $P$  the **defining polynomial** of the representation of  $H$  on  $\mathbb{C}^{h+1}$ . We will also use this name for the map  $P: \mathbb{C}^{h+1} \longrightarrow \mathbb{C}$  given by the same formula  $P(z) = \prod_j z_j^{\xi_j}$ , its extension to the local model  $P: Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0 \longrightarrow \mathbb{C}$  given by  $P([t, z, \nu]) = P(z)$ , and the induced quotient map  $\bar{P}: Y/T \longrightarrow \mathbb{C}$ . We trust that this will not cause confusion.

It is sometimes convenient to split a complexity one linear representation into the direct sum of two representations, one whose moment map is onto, and one which is toric.

**Lemma 5.13.** *Let an  $h$ -dimensional compact abelian Lie group  $H$  act on  $\mathbb{C}^{h+1}$  through an inclusion  $\rho: H \hookrightarrow (S^1)^{h+1}$ . After a permutation of the coordinates, there exist splittings*

$$H = H' \times H'' \quad \text{and} \quad \mathbb{C}^{h+1} = \mathbb{C}^{h'+1} \times \mathbb{C}^{h''},$$

*such that  $H'$  acts on  $\mathbb{C}^{h'+1}$  as a subgroup of  $(S^1)^{h'+1}$  with a surjective moment map, and  $H''$  acts on  $\mathbb{C}^{h''}$  through an isomorphism with  $(S^1)^{h''}$ .*

*Proof of Lemma 5.13.* Consider the defining polynomial,  $P(z) = \prod z_j^{\xi_j}$ . Let  $h''$  be the number of  $j$ 's such that  $\xi_j = 0$ , and let  $h' = h - h''$ . We can assume that  $\xi_j > 0$  for  $0 \leq j \leq h'$  and  $\xi_{h'+j} = 0$  for  $1 \leq j \leq h''$ . Then  $P$  defines a polynomial  $P': (S^1)^{h'+1} \rightarrow S^1$ .

Let us identify  $H$  with its image in  $(S^1)^{h+1}$ . Then

$$H = \ker P = \ker P' \times (S^1)^{h''}.$$

Let  $H' = \ker P'$  and  $H'' = (S^1)^{h''}$ . By Lemma 5.7, the moment map for  $H'$  is onto.  $\square$

## 6. THE TOPOLOGY OF THE QUOTIENT

In this section we describe the topology of the quotient  $M/T$ , in preparation for showing that two such quotients are  $\Phi$ -diffeomorphic if they have the same genus and isotropy data.

**Proposition 6.1.** *Let  $(M, \omega, \Phi, U)$  be a complexity one space.*

*The subset of  $M/T$  consisting of the complement of those moment fibers that contain single orbits is, topologically, a manifold with boundary.*

*The symplectic quotients  $\Phi^{-1}(\alpha)/T$  that contain more than one point are, topologically, closed connected oriented surfaces.*

*Proof.* The first claim follows immediately from Lemma 5.4, the local normal form theorem, and Lemma 6.2 below.

The fact that the symplectic quotients which contain more than one orbit are topological surfaces follows immediately from Lemma 5.4, the local normal form theorem, and Corollary 6.3 below. These surfaces are closed because the moment map is proper. They are connected by the connectedness of moment fibers. The symplectic structure on the symplectic quotient induces an orientation on the complement of a discrete set of points (namely, the exceptional orbits) and hence on the symplectic quotient itself.  $\square$

**Lemma 6.2.** *Let  $T$  be a torus. Let a closed  $h$ -dimensional subgroup  $H \subseteq T$  act on  $\mathbb{C}^{h+1}$  as a subgroup of  $(S^1)^{h+1}$  with a non-proper moment map and a defining polynomial  $P(z) = \prod z_j^{\xi_j}$ . Consider the model  $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$  and the map  $\overline{\Phi}_Y: Y/T \rightarrow \mathfrak{t}^*$  induced by the moment map. Define a map*

$$F: Y/T \rightarrow \mathfrak{t}^* \times \mathbb{C}$$

*by*

$$F := (\overline{\Phi}_Y, \overline{P}).$$

*Then  $F$  is a homeomorphism of  $Y/T$  with its image, which is the polygonal set  $(\text{image } \overline{\Phi}_Y) \times \mathbb{C}$ .*

**Corollary 6.3.** *The restriction of the defining polynomial to the symplectic quotient,  $\bar{P}_\alpha: \Phi_Y^{-1}(\alpha)/T \longrightarrow \mathbb{C}$ , is also a homeomorphism for all  $\alpha \in \text{image } \Phi_Y$ .*

**Definition 6.4.** Let  $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$  be a local model with a non-proper moment map. The map  $F$  of Lemma 6.2 is called the **trivializing homeomorphism** of the model.

The motivation for the name “trivializing homeomorphism” is that  $F$  exhibits the quotient  $Y/T$  as a trivial bundle, with fiber  $\mathbb{C}$ , over a polygonal subset of  $\mathfrak{t}^*$ . Moreover, once we remove the moment fibers which contain single orbits, the map  $\bar{\Phi}: M/T \longrightarrow U$  induced by the moment map exhibits  $M/T$ , topologically, as a surface bundle over image  $\Phi$ . This surface bundle plays an important role in the global classification of complexity one spaces, which will be given in subsequent papers.

*Proof of Lemma 6.2.* To show that  $F$  is a homeomorphism, it is both necessary and sufficient to prove that the map  $(\Phi_H, P): \mathbb{C}^{h+1} \longrightarrow (\text{image } \Phi_H) \times \mathbb{C}$  is onto and proper and that its fibers are exactly the  $H$ -orbits. (This follows from the formulas for  $F$  and  $\Phi_Y$ .)

We will begin by assuming that the moment map  $\Phi_H$  is onto  $\mathfrak{h}^*$ . By Lemma 5.7, this implies that the  $\xi_j$ ’s are positive.

Consider the commuting diagram

$$(6.5) \quad \begin{array}{ccc} \mathbb{C}^{h+1} & \xrightarrow{(\Phi_H, P)} & \mathfrak{h}^* \times \mathbb{C} \\ q_1 \downarrow & & \downarrow q_2 \\ \mathbb{R}_+^{h+1} & \xrightarrow{\bar{F}} & \mathfrak{h}^* \times \mathbb{R}_+, \end{array}$$

where

$$q_1(z_0, \dots, z_h) = (|z_0|^2, \dots, |z_h|^2), \quad q_2(\alpha, \zeta) = (\alpha, |\zeta|^2),$$

and

$$\bar{F}(x_0, \dots, x_h) = \left( \frac{1}{2} \sum_{j=0}^h x_j \eta_j, \prod_{j=0}^h x_j^{\xi_j} \right).$$

Let  $W$  be the boundary of the positive orthant  $\mathbb{R}_+^{h+1}$ . Since  $\xi_j > 0$  for all  $j$ , the map  $(x, t) \mapsto x + t\xi$  is a homeomorphism which identifies the product  $W \times \mathbb{R}_+$  with the orthant  $\mathbb{R}_+^{h+1}$ . The map  $\bar{F}$  then becomes a map from  $W \times \mathbb{R}_+$  to  $\mathfrak{h}^* \times \mathbb{R}_+$ , given by the formula

$$(x, t) \mapsto \left( \frac{1}{2} \sum_{j=0}^h x_j \eta_j, \prod_{j=0}^h (x_j + t\xi_j)^{\xi_j} \right),$$

where in the first coordinate we used the equality  $\sum \eta_i(x_i + t\xi_i) = \sum \eta_i x_i$ .

The map  $\bar{F}$  is one-to-one and onto, because the function  $x \mapsto \sum_{j=0}^h x_j \eta_j$  is a homeomorphism from  $W$  onto  $\mathfrak{h}^*$ , and because for each  $x \in W$ , the function

$$(6.6) \quad t \mapsto \prod_{j=0}^h (x_j + t\xi_j)^{\xi_j}$$

from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  is one to one and onto. The function (6.6) approaches infinity uniformly in  $x \in W$  as  $t \longrightarrow \infty$ . Therefore,  $\bar{F}$  is proper.

The properness of  $(\Phi_H, P)$  follows from that of  $\bar{F}$  and  $q_1$ .

Let us now show that  $(\Phi_H, P)$  is onto  $\mathfrak{h}^* \times \mathbb{C}$ . Since  $\bar{F}$  is onto, for any  $(\alpha, \zeta) \in \mathfrak{h}^* \times \mathbb{C}$  there exists  $z \in \mathbb{C}^{h+1}$  such that  $\Phi_H(z) = \alpha$  and  $|P(z)|^2 = |\zeta|^2$ . Choose  $b \in S^1$  so that  $P(z) = b\zeta$ . Since the map  $P: (S^1)^{h+1} \rightarrow S^1$  is onto, there exists  $a \in (S^1)^{h+1}$  such that  $P(a) = b^{-1}$ . Then  $(\Phi_H, P)(az) = (\alpha, \zeta)$ .

Let us now show that the level sets of  $(\Phi_H, P)$  are the orbits of  $H$ . Suppose that  $\Phi_H(z) = \Phi_H(z')$  and  $P(z) = P(z')$  for some  $z$  and  $z'$  in  $\mathbb{C}^{h+1}$ . Since  $\bar{F}$  is one to one, there exists  $\lambda \in (S^1)^{h+1}$  such that  $z' = \lambda z$ . We must show that  $\lambda$  can be chosen to be in  $H$ . If all the coordinates of  $z$  are non-zero,  $P(\lambda z) = P(z)$  implies that  $P(\lambda) = 1$ , which further implies that  $\lambda \in H$ , by (5.9).

If one of the coordinates of  $z$ , say  $z_0$ , is zero, then it is enough to show that the  $(S^1)^{h+1}$ -orbit of  $z$  coincides with the  $H$ -orbit of  $z$ . By a dimension count, it is enough to show that the  $(S^1)^{h+1}$ -stabilizer of  $z$  is not contained in  $H$ . Because  $z_0 = 0$ , the  $(S^1)^{h+1}$ -stabilizer of  $z$  contains the circle  $(a_0, 1, \dots, 1)$ . Since  $\xi_0 \neq 0$ , the polynomial  $P$  is not constant on this circle. By exactness of (5.9), this circle is not contained in  $H$ .

For the general case, we may let

$$\mathbb{C}^{h+1} = \mathbb{C}^{h'+1} \times \mathbb{C}^{h''} \quad \text{and} \quad H = H' \times H''$$

be the splitting into a surjective part and a toric part, as described in Lemma 5.13. Then  $\Phi_H(z, w) = (\Phi_{H'}(z), \Phi_{H''}(w))$ . The map  $z \mapsto (\Phi_{H'}(z), P(z))$  is proper, its fibers are the  $H'$  orbits, and it is onto  $(\mathfrak{h}')^* \times \mathbb{C}$ , as we have shown above. The map  $w \mapsto \Phi_{H''}(w)$  is a moment map for a toric action, so it is proper and its level sets are  $H''$  orbits. Thus, the map

$$(\Phi_H, P): (z, w) \mapsto (\Phi_{H'}(z), \Phi_{H''}(w), P(z))$$

is proper, onto  $(\text{image } \Phi_H) \times \mathbb{C}$ , and its level sets are the  $H$ -orbits. This is precisely what we needed in order to deduce that  $F$  is a homeomorphism.  $\square$

## 7. THE SMOOTH STRUCTURE ON THE QUOTIENT.

In the previous section we showed that, if no moment fiber contains exactly one orbit, both the ordinary quotient  $M/T$  and the symplectic quotient  $\Phi^{-1}(\alpha)/T$  are topologically manifolds (with boundary). In this section we will show that they are smooth manifolds (with corners) — outside the exceptional orbits.

We have already given  $M/T$  the quotient differentiable structure, by specifying the sheaf of smooth functions to be the sheaf of functions that pull back to smooth  $T$ -invariant functions on  $M$  (as in section 4). However, there is another way to give  $M/T$  a differentiable structure: we can cover  $M$  with local models,  $Y$ , and take the trivializing homeomorphisms  $F: Y/T \rightarrow (\text{image } \Phi_Y) \times \mathbb{C}$  (see section 6) as local charts. Outside the set of exceptional orbits, the two strategies give the same well-defined smooth structure:

**Lemma 7.1.** *Let  $T$  be a torus. Let a closed  $h$ -dimensional subgroup  $H$  of  $T$  act on  $\mathbb{C}^{h+1}$  as a subgroup of  $(S^1)^{h+1}$  with a non-proper moment map. Consider the model  $Y = T \times_H \mathbb{C}^{h+1} \times \mathfrak{h}^0$ . Let  $E \subset Y$  be the union of the exceptional orbits. Let*

$$F = (\bar{\Phi}_Y, \bar{P}): Y/T \rightarrow \mathfrak{t}^* \times \mathbb{C}$$

*be the trivializing homeomorphism (see Definition 6.4).*

*Then the restriction of  $F$  to  $(Y \setminus E)/T$  pulls back the sheaf of smooth functions on  $\mathfrak{t}^* \times \mathbb{C}$  onto the sheaf of smooth functions on the quotient.*



A similar statement holds for the symplectic quotients:

**Corollary 7.2.** *The restriction of the defining polynomial*

$$\overline{P}_\alpha: (\Phi_Y^{-1}(\alpha) \cap (Y \setminus E))/T \longrightarrow \mathbb{C}$$

*is a diffeomorphism onto its image.*

In contrast, at the exceptional orbits the trivializing homeomorphism and defining polynomial need not respect the differentiable structure.

Our proof of Lemma 7.1 will use the following criterion for non-exceptional orbits:

**Lemma 7.3.** *Let an  $h$  dimensional compact abelian Lie group  $H$  act on  $\mathbb{C}^{h+1}$  as a subgroup of  $(S^1)^{h+1}$  with a surjective moment map. Let  $P(z) = \prod z_j^{\xi_j}$  be the defining polynomial. The orbit of  $z \in \mathbb{C}^{h+1}$  is exceptional unless*

1.  $z_j \neq 0$  for all  $j$ , or
2. there exists an index  $i$  such that  $\xi_i = 1$  and  $z_j \neq 0$  for all  $j \neq i$ .

*Proof.* Since the moment map is onto, by Corollary 4.7 the  $H$ -orbit of  $z$  is non-exceptional if and only if the stabilizer of  $z$  in  $H$  is trivial. We identify  $H$  with the subgroup of  $(S^1)^{h+1}$  by which it acts. The stabilizer of  $z$  then consists of those elements  $\lambda \in (S^1)^{h+1}$  such that  $\lambda_j = 1$  whenever  $z_j \neq 0$  and such that  $P(\lambda) = \prod \lambda_j^{\xi_j} = 1$ .  $\square$

Before proving Lemma 7.1 in its full generality, we prove the following variant, which in particular implies the lemma for the case that the moment map is onto.

**Lemma 7.4.** *Let a compact abelian group  $H$  act on  $\mathbb{C}^{h+1}$  as a codimension one subgroup of  $(S^1)^{h+1}$  with a surjective moment map  $\Phi_H$ . Denote by  $U$  the union of the non-exceptional orbits in  $\mathbb{C}^{h+1}$ . For any manifold  $N$ , the map*

$$(7.5) \quad (id, \overline{\Phi}_H, \overline{P}): N \times (U/H) \longrightarrow N \times \mathfrak{h}^* \times \mathbb{C},$$

*given by*

$$(n, [z]) \mapsto (n, \Phi_H(z), P(z)),$$

*is a diffeomorphism with its image, i.e., it pulls back the sheaf of smooth functions on  $N \times \mathfrak{h}^* \times \mathbb{C}$  onto the sheaf of smooth functions on the quotient.*

*Proof.* Since  $H$  acts freely on  $U$ , the quotient  $U/H$  is naturally a smooth manifold (with the quotient differential structure). The map (7.5) is smooth, and, by Lemma 6.2, it is a homeomorphism onto its image. Since a smooth homeomorphism between two smooth manifolds of the same dimension is a diffeomorphism exactly at those points where it is a submersion, we need to show that  $(d\Phi_H, dP)|_z$  is onto for all non-exceptional  $z \in \mathbb{C}^{h+1}$ .

To show that  $(d\Phi_H, dP)|_z$  is onto, it is enough to find  $\zeta \in T_z \mathbb{C}^n = \mathbb{C}^n$  such that  $d\Phi_H|_z(\zeta) = d\Phi_H(\sqrt{-1}\zeta) = 0$  and  $dP|_z(\zeta) \neq 0$ . To see this note that, since  $H$  acts freely,  $d\Phi_H|_z$  is onto  $\mathfrak{h}^*$ . Additionally, since  $P$  is holomorphic,  $dP|_z(\zeta)$  and  $dP|_z(\sqrt{-1}\zeta) = \sqrt{-1}dP|_z(\zeta)$  form a real basis to  $\mathbb{C}$ .

Recall that  $P(z) = \prod z_j^{\xi_j}$  and  $\Phi_H(z) = \frac{1}{2} \sum_j \eta_j z_j \overline{z}_j$ . Hence

$$d\Phi_H|_z(\zeta) = \frac{1}{2} \sum_j \eta_j (z_j \overline{\zeta}_j + \zeta_j \overline{z}_j).$$

*Subcase A: all the coordinates of  $z$  are non-zero.* In this case,

$$dP|_z(\zeta) = P(z) \sum_j \frac{\xi_j}{z_j} \zeta_j.$$

Let  $(\zeta_j) = (\frac{\xi_j}{\bar{z}_j})$ . Then

$$\begin{aligned} d\Phi_H|_z(\zeta) &= \frac{1}{2} \sum_j \eta_j \left( z_j \frac{\xi_j}{z_j} + \frac{\xi_j}{\bar{z}_j} \bar{z}_j \right) \\ &= \frac{1}{2} \sum_j \eta_j (\xi_j + \xi_j) = 0 \end{aligned}$$

by (5.11), and

$$\begin{aligned} d\Phi_H(\sqrt{-1}\zeta) &= \frac{1}{2} \sum_j \eta_j \left( z_j (-\sqrt{-1} \frac{\xi_j}{z_j}) + \sqrt{-1} \frac{\xi_j}{\bar{z}_j} \bar{z}_j \right) \\ &= \frac{1}{2} \sum_j \eta_j (-\sqrt{-1} \xi_j + \sqrt{-1} \xi_j) = 0, \end{aligned}$$

whereas

$$dP|_z(\zeta) = P(z) \sum_j \frac{\xi_j}{z_j} \frac{\xi_j}{\bar{z}_j} \neq 0.$$

*Subcase B: one of the coordinates of  $z$ , say,  $z_1$ , is zero,  $\xi_1 = 1$ , and  $z_j \neq 0$  for all  $j \neq 1$ .* In this case,

$$dP|_z(\zeta) = \left( \prod_{j \neq 1} z_j^{\xi_j} \right) \zeta_1.$$

Let  $\zeta_1 = 1$  and  $\zeta_j = 0$  for all  $j \neq 1$ . Then

$$d\Phi_H|_z(\zeta) = \frac{1}{2} \eta_1 (z_1 + \bar{z}_1) = 0,$$

and

$$d\Phi_H|_z(\sqrt{-1}\zeta) = \frac{1}{2} \eta_1 (-\sqrt{-1} z_1 + \sqrt{-1} \bar{z}_1) = 0,$$

whereas

$$dP|_z(\zeta) = \left( \prod_{j \neq 1} z_j^{\xi_j} \right) \neq 0.$$

□

*Proof of Lemma 7.1.* Let  $\mathbb{C}^{h+1} = \mathbb{C}^{h'+1} \times \mathbb{C}^{h''}$  and  $H = H' \times H''$  be the splitting into a surjective part and a toric part, as described in Lemma 5.13. With this splitting, the local model is

$$Y = T \times_{H'} \mathbb{C}^{h'+1} \times_{H''} \mathbb{C}^{h''} \times \mathfrak{h}^0,$$

and its quotient is

$$Y/T = (\mathbb{C}^{h'+1}/H') \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0.$$

The union of the non-exceptional orbits in this quotient is

$$(7.6) \quad (U'/H') \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0,$$

where  $U'$  is the union of the free orbits in  $\mathbb{C}^{h'+1}$ . Under the identification  $\mathfrak{t}^* = (\mathfrak{h}')^* \times (\mathfrak{h}'')^* \times \mathfrak{h}^0$ , the trivializing homeomorphism  $F$  on (7.6) is

$$F([z'], [z''], \nu) = (\Phi_{H'}(z'), \Phi_{H''}(z''), \nu, P(z')),$$

where  $P$  is the defining polynomial.

Lemma 7.4 implies that the map

$$([z'], [z''], \nu) \mapsto (\Phi_{H'}(z'), [z''], \nu, P(z'))$$

pulls back the sheaf of smooth functions on  $(\mathfrak{h}')^* \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0 \times \mathbb{C}$  onto the sheaf of smooth functions on  $(U'/H') \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0$ . Therefore, it is enough to show that the map

$$(7.7) \quad (\alpha, [z''], \nu, \zeta) \mapsto (\alpha, \Phi_{H''}(z''), \nu, \zeta)$$

pulls back the sheaf of smooth functions on  $(\mathfrak{h}')^* \times (\mathfrak{h}'')^* \times \mathfrak{h}^0 \times \mathbb{C}$  onto the sheaf of smooth functions on  $(\mathfrak{h}')^* \times (\mathbb{C}^{h''}/H'') \times \mathfrak{h}^0 \times \mathbb{C}$ .

By a theorem of Schwartz [Sch1], any invariant smooth function can be expressed as a smooth function of real invariant polynomials. Since  $H''$  acts on  $\mathbb{C}^{h''}$  through an isomorphism with  $(S^1)^{h''}$ , the ring of  $H''$ -invariant polynomials in  $(\alpha, z'', \nu, \zeta)$  is generated by the coordinates of  $\alpha$  and  $\nu$ , the real and imaginary parts of  $\zeta$ , and  $|z_1|^2, \dots, |z_{h''}|^2$ . Finally, note that

$$\Phi_{H''}(z_1, \dots, z_{h''}) = A(|z_1|^2, \dots, |z_{h''}|^2),$$

where  $A: \mathbb{R}^{h''} \rightarrow (\mathfrak{h}'')^*$  is the linear isomorphism dual to the map  $H'' \rightarrow (S^1)^{h''}$ . Hence, every smooth invariant function is the pullback via (7.7) of a smooth function.  $\square$

## 8. THE ASSOCIATED SURFACE

In this section we associate to a moment fiber  $\Phi^{-1}(\alpha)$  in a complexity one space a smooth **marked surface** whose underlying topological space is the symplectic quotient  $\Phi^{-1}(\alpha)/T$ .

We do not, however, define a functor from a category of complexity one spaces to a category of marked surfaces. For one thing, our construction depends on a choice. More seriously, a smooth map between two complexity one spaces does not induce a smooth map between the associated surfaces, nor visa versa. This is unfortunate, because we will easily obtain an isomorphism between the marked surfaces if the genus and isotropy data are the same. In the next few sections we will show that, in spite of this lack of functoriality, we can obtain an isomorphism of complexity one spaces from an isomorphism of marked surfaces.

As we saw in the previous section, on the complement of the exceptional orbits the symplectic quotient is naturally a smooth surface. Unfortunately, it is not naturally smooth near the exceptional orbits. Nevertheless, we can use the defining polynomials of section 5 to give it a smooth structure. However, in order to do this, we must make arbitrary choices: we must identify open subsets of the manifold with open subsets of the local models. We call these choices grommets:

**Definition 8.1.** Let  $(M, \omega, \Phi, U)$  be a complexity one space. A **grommet** is a  $\Phi$ - $T$ -diffeomorphism  $\psi: D \rightarrow M$  from an open subset  $D$  of a local model  $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$  onto an open subset of  $M$ .

The name “grommet” is better motivated in the following definition:

**Definition 8.2.** Let  $\Sigma$  be a smooth oriented two dimensional manifold. A **grommet** at a point  $q \in \Sigma$  is a diffeomorphism  $\varphi: B \rightarrow \Sigma$  from a neighborhood  $B$  of the origin in  $\mathbb{C}$  onto an open subset of  $\Sigma$ , such that  $\varphi$  sends the origin 0 to the point  $q$ .

*Remark 8.3.* Let  $(M, \omega, \Phi, U)$  be a complexity one space and consider  $\alpha \in U$  such that the moment fiber  $\Phi^{-1}(\alpha)$  contains more than one orbit. Any grommet

$\psi: D \rightarrow M$  with  $\psi([t, 0, 0]) = \mathcal{O} \in \Phi^{-1}(\alpha)$  induces a “coordinate chart” on the symplectic quotient  $\Phi^{-1}(\alpha)/T$ , that is, a homeomorphism  $\varphi$  from a subset  $B \subset \mathbb{C}$  into  $\Phi^{-1}(\alpha)/T$ , such that  $\varphi(0) = \mathcal{O}$ . Explicitly, the map  $\overline{P}_\alpha: (D \cap \Phi_Y^{-1}(\alpha))/T \rightarrow \mathbb{C}$  given by the defining polynomial is a homeomorphism onto its image  $B$ , and  $\varphi := \overline{\psi} \circ \overline{P}_\alpha^{-1}: B \rightarrow \Phi^{-1}(\alpha)/T$  is a homeomorphism onto its image, where  $\overline{\psi}$  is induced from  $\psi$ .

**Definition 8.4.** Let  $(M, \omega, \Phi, U)$  be a complexity one space and  $\alpha \in U$  a point whose moment fiber contains more than one orbit. For each exceptional orbit  $E_j$  in  $\Phi^{-1}(\alpha)$ , let  $\psi_j: D_j \rightarrow M$  be a grommet such that  $\psi_j([t, 0, 0]) = E_j$ . The **associated marked surface** consists of the following data:

1. The connected oriented two-dimensional topological manifold  $\Sigma = M_{\text{red}} = \Phi^{-1}(\alpha)/T$ .
2. The set of marked points  $\{q_j\}$  in  $\Sigma$  that corresponds to the set of exceptional orbits  $\{E_j\}$  in  $\Phi^{-1}(\alpha)$ .
3. The smooth manifold structure on  $\Sigma$  that is given by the following coordinate charts. For each exceptional orbit  $E_j$  in  $\Phi^{-1}(\alpha)$ , take the given grommet. For each non-exceptional orbit  $\mathcal{O}$  in  $\Phi^{-1}(\alpha)$ , choose an arbitrary grommet with  $\psi([t, 0, 0]) = \mathcal{O}$ . For each grommet, take the induced coordinate chart on  $\Sigma$  as described in Remark 8.3.
4. At each marked point  $q_j$ , the grommet on  $\Sigma$  that is given by the coordinate chart of item 3.
5. For each marked point  $q_j$ , a label consisting of the isotropy representation at the corresponding exceptional orbit.

The image of the grommet  $\psi_j$  does not contain any of the other exceptional orbits,  $E_i$ ,  $i \neq j$ ; this follows from the fact that the model contains at most one exceptional orbit in each moment fiber (see Lemma 7.3). The fact that the charts in item 3 give a well defined smooth structure on  $M/T$  follows from this and from the fact that the smooth structures coincide on this complement (see Corollary 7.2).

## 9. FLATTENING THE QUOTIENT

We are now ready to show that, after possibly replacing  $M$  by the preimage of a small subset of  $\mathfrak{t}^*$ , the quotient  $M/T$  is determined by the associated marked surface  $\Sigma$ . Topologically, there is a homeomorphism from the quotient  $M/T$  to the product  $\Sigma \times (\text{image } \Phi)$ . Moreover, this homeomorphism can be chosen to respect the smooth structure in a certain sense. Such a homeomorphism is called a flattening; a precise definition is given below.

We begin by the definition for a local model. Let  $T$  be a torus, and let a closed subgroup  $H \subseteq T$  act on  $\mathbb{C}^n$  as a codimension one subgroup of  $(S^1)^n$  with a non-proper moment map. Consider the model

$$Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0,$$

with moment map  $\Phi_Y: Y \rightarrow \mathfrak{t}^*$ . Recall from Corollary 6.3 that the defining polynomial  $\overline{P}: Y/T \rightarrow \mathbb{C}$  restricts to a homeomorphism  $\overline{P}_\alpha: \Phi_Y^{-1}(\alpha)/T \rightarrow \mathbb{C}$ .

**Definition 9.1.** The **standard flattening** of  $Y$  is the map

$$\delta: Y/T \rightarrow (\Phi_Y^{-1}(\alpha)/T) \times (\text{image } \Phi_Y)$$

given by

$$\delta := ((\overline{P}_\alpha^{-1} \circ \overline{P}), \overline{\Phi}_Y).$$

The standard flattening is a homeomorphism by Lemma 6.2.

**Definition 9.2.** The **exceptional sheet** in the model  $Y$  is the subset

$$S := \{[t, z, \nu] \in T \times_H \mathbb{C}^n \times \mathfrak{h}^0 \mid P(z) = 0\}.$$

Every exceptional orbit is contained in  $S$ , by Lemma 7.3. Thus, by Lemma 7.1 and Corollary 7.2, the standard flattening  $\delta$  is a diffeomorphism of  $(Y \setminus S)/T$  with its image.

Grommets were defined in Definition 8.1. Flattenings will involve grommets that are sufficiently large, in the following sense:

**Definition 9.3.** Let  $(M, \omega, \Phi, U)$  be a complexity one space. A grommet  $\psi: D \rightarrow M$  is **wide** if  $D$  is an open subset of a model  $Y$  on which the moment map  $\Phi_Y$  is non-proper and if the domain  $D$  contains that part of the exceptional sheet that lies over  $U$ , i.e., if  $(\Phi_Y^{-1}(U) \cap S) \subset D$ .

We are now ready to define the flattening of a complexity one space.

**Definition 9.4.** Let  $(M, \omega, \Phi, U)$  be a complexity one space. Assume that the moment fiber over  $\alpha \in U$  contains more than one orbit. A **flattening** of the space  $(M, \omega, \Phi)$  about  $\alpha$  consists of the following data.

1. A homeomorphism

$$(9.5) \quad \delta: M/T \rightarrow (\Phi^{-1}(\alpha)/T) \times (\text{image } \Phi)$$

whose second coordinate is induced by the moment map.

2. For each exceptional orbit  $E_j$  in  $\Phi^{-1}(\alpha)$ , a wide grommet  $\psi_j: D_j \rightarrow M$  such that  $\psi_j([t, 0, 0]) = E_j$ .

We require that the following two conditions be satisfied:

1. The restriction of  $\delta$  to the complement of the exceptional sheets,

$$\delta: M/T \setminus \sqcup_j \psi_j(S_j \cap D_j)/T \rightarrow (\Phi^{-1}(\alpha) \setminus \sqcup_j E_j)/T \times (\text{image } \Phi),$$

must be a diffeomorphism, in the sense discussed at the beginning of section 4.

2. Additionally, near the exceptional sheets  $\delta$  must be given by the standard flattenings of the local models. More precisely, the following diagram must commute:

$$(9.6) \quad \begin{array}{ccc} D_j/T & \xrightarrow{\delta_j} & (\Phi_j^{-1}(\alpha) \cap D_j)/T \times \mathfrak{t}^* \\ \downarrow \overline{\psi}_j & & \downarrow (\overline{\psi}_j, \text{id}) \\ M/T & \xrightarrow{\delta} & (\Phi^{-1}(\alpha)/T) \times \mathfrak{t}^*, \end{array}$$

where  $\Phi_j$  is the moment map on the corresponding local model  $Y_j \supset D_j$ , where  $\delta_j: Y_j/T \rightarrow (\Phi_j^{-1}(\alpha)/T) \times (\text{image } \Phi_j)$  denotes the standard flattening of  $Y_j$ , and where  $\overline{\psi}_j: D_j/T \rightarrow M/T$  is induced by the grommet. In particular, we require that the diagram be well defined, i.e., that the image  $\delta_j(D_j/T)$  be contained in  $((\Phi_j^{-1}(\alpha) \cap D_j)/T) \times \mathfrak{t}^*$ .

The rest of this section is devoted to showing that flattenings always exist locally:

**Lemma 9.7.** *Let  $(M, \omega, \Phi, U)$  be a complexity one space, and let  $\alpha \in \mathfrak{t}^*$  be a point whose moment fiber contains more than one orbit. Then there exists a neighborhood  $V$  of  $\alpha$  contained in  $U$  whose preimage,  $\Phi^{-1}(V)$ , admits a flattening about  $\alpha$ .*

Since image  $\Phi$  is convex, and hence contractible, the following consequence is immediate:

**Corollary 9.8.** *Let  $(M, \omega, \Phi, U)$  be a complexity one space. Consider  $\alpha \in U$  such that the moment fiber  $\Phi^{-1}(\alpha)$  contains more than one orbit. Then for every sufficiently small convex neighborhood  $V$  of  $\alpha$ , the restriction map*

$$H^*(V/T) \longrightarrow H^*(\Phi^{-1}(y)/T)$$

*is an isomorphism for all  $y \in \Phi(V)$ . In particular,  $\Phi^{-1}(V)$  satisfies Condition (3.2).*

Since the set of points in  $U$  whose moment fiber has more than one orbit is connected by Lemma 5.4, Lemma 9.7 has the following additional immediate consequence:

**Corollary 9.9.** *Let  $(M, \omega, \Phi, U)$  be a complexity one space. Then all the symplectic quotients  $\Phi^{-1}(\alpha)/T$  that contain more than one point have the same genus. Thus, the genus of a complexity one space (see section 1) is well-defined.*

The following lemma is a first step for the proof of Lemma 9.7.

**Lemma 9.10.** *Let  $(M, \omega, \Phi, U)$  be a complexity one space, and assume the moment fiber over  $\alpha \in U$  contains more than one orbit. Denote the exceptional orbits in  $\Phi^{-1}(\alpha)$  by  $\{E_j\}$ .*

*After replacing  $M$  by the preimage of some neighborhood of  $\alpha$  in  $U$ , there exist wide grommets  $\psi_j: D_j \longrightarrow M$  such that  $\psi_j([t, 0, 0]) = E_j$  and the images  $\psi_j(D_j)$  have pairwise disjoint closures.*

*Proof.* By Lemma 5.4, for every exceptional orbit  $E_j$  over  $\alpha$ , the corresponding local model has a non-proper moment map,  $\Phi_j: Y_j \longrightarrow \mathfrak{t}^*$ . By the local normal form theorem, we may choose a grommet  $\psi_j: D_j \longrightarrow M$  such that  $\psi_j([t, 0, 0]) = E_j$ .

By Lemma 6.2 and Definition 9.2, the moment map  $\Phi_j$  restricts to a homeomorphism of the exceptional sheet  $S_j \subset Y_j$  with the image of  $\Phi_j$ . Hence there exists a neighborhood  $W_j$  of  $\alpha$  such that  $S_j \cap D_j = S_j \cap \Phi_j^{-1}(W_j)$ .

For  $i \neq j$ , the intersection  $\psi_i(S_i \cap D_i) \cap \psi_j(S_j \cap D_j)$  is a closed subset of  $M$  which does not meet the fiber  $\Phi^{-1}(\alpha)$ . Since the moment map is proper, there exists a neighborhood  $V$  of  $\alpha$  which does not meet the image under the moment map of any of these intersections.

If we define  $W := \bigcap_j W_j \cap V$  of  $\alpha$ , and replace  $M$  by  $M \cap \Phi^{-1}(W)$  and  $D_j$  by  $D_j \cap \Phi_j^{-1}(W)$ , the grommets  $\psi_j$  become wide. Also, the exceptional sheets  $\psi_j(S_j \cap D_j)$  are then closed and disjoint, so we can shrink each  $D_j$  to a smaller neighborhood of  $S_j \cap D_j$  to obtain wide grommets whose images have pairwise disjoint closures.  $\square$

*Proof of Lemma 9.7.* Let  $\psi_j: D_j \longrightarrow M$  be wide grommets such that  $\psi_j([t, 0, 0]) = E_j$  are the exceptional orbits in the moment fiber  $\Phi^{-1}(\alpha)$  and such that the images  $\psi_j(D_j)$  have disjoint closures in  $M$ . These grommets exist by Lemma 9.10. This will not be ruined if we further restrict to a smaller neighborhood of  $\alpha$ .

Recall that the standard flattening of the local model  $Y_j$  is

$$\delta_j = (g_j, \bar{\Phi}_j): Y_j/T \longrightarrow (\Phi_j^{-1}(\alpha)/T) \times (\text{image } \Phi_j)$$

where  $g_j = (\bar{P}_{j,\alpha})^{-1} \circ \bar{P}_j$ . Replace  $D_j/T$  by its intersection with  $g_j^{-1}((\Phi_j^{-1}(\alpha) \cap D_j)/T)$ . Then the restriction

$$\delta_j: D_j/T \longrightarrow ((\Phi_j^{-1}(\alpha) \cap D_j)/T) \times (\text{image } \Phi_j)$$

is well defined. After this, the grommets determine a unique map  $\delta$  on the images of  $D_j/T$  in  $M/T$  such that the following diagram commutes.

$$(9.11) \quad \begin{array}{ccc} D_j/T & \xrightarrow{\delta_j} & ((\Phi_j^{-1}(\alpha) \cap D_j)/T) \times \mathfrak{t}^* \\ \downarrow \bar{\psi}_j & & \downarrow (\bar{\psi}_j, \text{id}) \\ \sqcup_j \bar{\psi}_j(D_j/T) & \xrightarrow{\delta} & (\Phi^{-1}(\alpha)/T) \times \mathfrak{t}^*. \end{array}$$

We need to extend  $\delta$  to the rest of  $M/T$ , perhaps after shrinking the  $D_j$ s to smaller neighborhoods of  $S_j \cap \Phi_j^{-1}(U)$ .

Using the stability of the moment map, Lemma 7.1 implies that on the complement of the exceptional sheets in the quotient  $M/T$ , the map  $\bar{\Phi}: M/T \setminus \sqcup_j \psi_j(S_j \cap D_j)/T \longrightarrow (\text{image } \Phi)$  induced by the moment map is a submersion. Namely, for each point  $[m]$  in the domain of this map there exists a neighborhood  $W$  of  $\Phi(m)$  in  $\mathfrak{t}^*$  such that a neighborhood of  $[m]$  is diffeomorphic to the product of a disk with  $W \cap (\text{image } \Phi)$  with the map  $\bar{\Phi}$  being the projection map.

The partial flattening (9.11) determines an Ehresmann connection for this submersion, defined on the open subset  $\sqcup_j \psi_j(D_j \setminus S_j)/T$ : we declare the horizontal tangent vectors to be those whose push-forward by  $\delta$  is tangent to the sheets  $\{q\} \times \mathfrak{t}^*$  for  $q \in \Phi^{-1}(\alpha)/T$ .

We extend this to an Ehresmann connection on the entire complement of the exceptional sheets,  $M/T \setminus \sqcup_j \psi_j(S_j \cap D_j)/T$ , perhaps after shrinking the  $D_j$ s; this is easily done with a partition of unity. Then for a point  $p$  in  $M/T \setminus \sqcup_j \psi_j(S_j \cap D_j)/T$ , any path  $\gamma$  in  $U$  which starts at  $\bar{\Phi}(p)$  can be lifted to a horizontal path in  $M/T \setminus \sqcup_j \psi_j(S_j \cap D_j)/T$ .

We proceed as in the proof of Ehresmann's lemma. Let us assume that  $\alpha = 0$  and that  $U$  is a ball centered at 0. We can choose coordinates on  $\mathfrak{t}^*$  such that  $(\text{image } \Phi)$  becomes

$$(\text{image } \Phi) = U \cap (\mathbb{R}^k \times \mathbb{R}_+^l), \quad k + l = m = \dim \mathfrak{t}^*.$$

(This is possible by Lemma 4.6.) Denote by  $v_1, \dots, v_m$  the standard vector fields on  $\mathfrak{t}^*$  that are parallel to the coordinate axes, let  $\tilde{v}_1, \dots, \tilde{v}_m$  be their horizontal liftings to  $M/T \setminus \sqcup_j \psi_j(S_j \cap D_j)/T$ , and let  $f_j^t$ , for  $i = 1, \dots, m$  and  $t \in \mathbb{R}$ , be the flows which the  $\tilde{v}_i$  generate. For  $p \in M/T$ , define

$$\delta(p) = (g(p), \bar{\Phi}(p))$$

where, if  $(t_1, \dots, t_m)$  are the coordinates of  $p$ , then  $g(p) \in \Phi^{-1}(\alpha)/T$  is given by  $g(p) = f_1^{-t_1} \dots f_m^{-t_m}(p)$ .  $\square$

## 10. DIFFEOMORPHISM BETWEEN QUOTIENTS

In this section, we show that if two complexity one spaces equipped with flattenings have the same genus and isotropy data, their quotients are  $\Phi$ -diffeomorphic, in the sense of Definition 4.1:

**Proposition 10.1.** *Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces that admit flattenings about a point  $\alpha \in U$ . If the spaces have the same genus and isotropy data, then there exists a  $\Phi$ -diffeomorphism from  $M/T$  to  $M'/T$ .*

**Definition 10.2.** Let  $\Sigma$  and  $\Sigma'$  be closed oriented surfaces with labeled marked points and with grommets at these points. (See Definition 8.2.) An orientation preserving diffeomorphism  $g: \Sigma \rightarrow \Sigma'$  is **rigid** if

- it induces a bijection between the marked points in  $\Sigma$  and those in  $\Sigma'$ , and sends each marked point  $q_j$  in  $\Sigma$  to a marked point  $q'_j$  in  $\Sigma'$  with the same label;
- for each marked point  $q_j \in \Sigma$  and  $q'_j \in \Sigma'$  and corresponding grommets  $\varphi_j$  and  $\varphi'_j$ , the composition

$$\varphi'^{-1}_j \circ g \circ \varphi_j,$$

which is a diffeomorphism from a neighborhood of 0 in  $\mathbb{C}$  to another neighborhood of 0 in  $\mathbb{C}$ , coincides with a rotation of  $\mathbb{C}$  on some (smaller) neighborhood of 0.

*Proof of Proposition 10.1.* Let  $(M, \omega, \Phi, U)$  and  $(M', \omega', \Phi', U)$  be complexity one spaces with flattenings about the point  $\alpha \in U$ . Let  $\Sigma$  and  $\Sigma'$  be the associated marked surfaces, as in Definition 8.4.

If the spaces have the same isotropy data, there exists a bijection from the marked points in  $\Sigma$  onto the marked points in  $\Sigma'$  which respects the isotropy labels. If the spaces have the same genus, this bijection extends to an orientation preserving diffeomorphism from  $\Sigma$  to  $\Sigma'$ ; this follows from standard differential topology. Moreover, this diffeomorphism can be deformed near the marked points into a rigid diffeomorphism  $h: \Sigma \rightarrow \Sigma'$ . This type of result is standard in differential topology; see, e.g., [Ko, II, 5.2].

The images of the moment maps  $\Phi$  and  $\Phi'$  are the same; this follows from Lemma 4.6 and from the fact that the isotropy data are the same. Let

$$\delta: M/T \rightarrow \Phi^{-1}(\alpha)/T \times (\text{image } \Phi)$$

and

$$\delta': M'/T \rightarrow \Phi'^{-1}(\alpha)/T \times (\text{image } \Phi')$$

be maps given in the flattenings. Under the identification of the symplectic quotients  $\Phi^{-1}(\alpha)/T$  and  $\Phi'^{-1}(\alpha)/T$  with  $\Sigma$  and  $\Sigma'$ , respectively, the rigid diffeomorphism  $h: \Sigma \rightarrow \Sigma'$  extends to a map

$$(h, \text{id}): (\Phi^{-1}(\alpha)/T) \times (\text{image } \Phi) \rightarrow (\Phi'^{-1}(\alpha)/T) \times (\text{image } \Phi').$$

We will show that the map  $g: M/T \rightarrow M'/T$  defined by

$$g := \delta'^{-1} \circ (h, \text{id}) \circ \delta$$

is a  $\Phi$ -diffeomorphism.

The diffeomorphism  $h$  fixes an identification between exceptional orbits in  $\Phi^{-1}(\alpha)$  and  $\Phi'^{-1}(\alpha)$  with the same isotropy representation. Thus, we can unequivocally denote by  $\{Y_j\}$  the local models for the exceptional orbits over  $\alpha$  in both  $M$  and  $M'$ . Let

$$\psi_j: D_j \rightarrow M \quad \text{and} \quad \psi'_j: D'_j \rightarrow M'$$



denote the grommets, with  $D_j \subseteq Y_j$  and  $D'_j \subseteq Y'_j$ , and let  $E_j$  and  $E'_j$  denote the exceptional orbits in  $\Phi^{-1}(\alpha)$  and in  $\Phi'^{-1}(\alpha)$ .

Our first claim is that the restriction

$$g: (M \setminus \sqcup_j \psi_j(S_j \cap D_j)) / T \longrightarrow (M' \setminus \sqcup_j \psi'_j(S_j \cap D_j)) / T$$

is a  $\Phi$ -diffeomorphism. This is easy: by the definition of flattening, the restrictions

$$\delta: (M \setminus \sqcup_j \psi_j(S_j \cap D_j)) / T \longrightarrow (\Phi^{-1}(\alpha) \setminus \sqcup_j E_j) / T \times (\text{image } \Phi)$$

and

$$\delta': (M' \setminus \sqcup_j \psi'_j(S_j \cap D_j)) / T \longrightarrow (\Phi'^{-1}(\alpha) \setminus \sqcup_j E'_j) / T \times (\text{image } \Phi')$$

are both diffeomorphisms. Moreover, the map

$$(\Phi^{-1}(\alpha) \setminus \sqcup_j E_j) / T \longrightarrow (\Phi'^{-1}(\alpha) \setminus \sqcup_j E'_j) / T$$

induced by  $h$  is a diffeomorphism, since the smooth structures on  $\Phi^{-1}(\alpha)/T$  and  $\Sigma$  agree off the exceptional orbits.

It remains to show that  $g$  is a  $\Phi$ -diffeomorphism in a neighborhood of each exceptional sheet  $\psi_j(S_j \cap D_j)/T$ .

Let  $\varphi_j: B_j \longrightarrow \Sigma$  and  $\varphi'_j: B'_j \longrightarrow \Sigma'$  denote the grommets of the associated surfaces. Since  $h$  is rigid, there exist  $a_j \in S^1$  such that  $\varphi_j^{-1} \circ h \circ \varphi_j$  is given by rotation by  $a_j \in S^1$  on some neighborhood of the origin in  $\mathbb{C}$ .

Let  $P_j: (S^1)^{n_j} \longrightarrow S^1$  be the defining polynomial for the exceptional orbit  $E_j$ . Since  $P_j$  is surjective, we may choose  $\lambda_j \in (S^1)^{n_j}$  so that  $P_j(\lambda_j) = a_j$ . This defines an equivariant symplectomorphism from the local model  $Y_j = T \times_{H_j} \mathbb{C}^{n_j} \times \mathfrak{h}_j^0$  to itself as follows:

$$(10.3) \quad \lambda_j \cdot ([t, z, \nu]) = [t, \lambda_j \cdot z, \nu].$$

This map induces a  $\Phi$ -diffeomorphism on the quotient,  $g_j: Y_j/T \longrightarrow Y_j/T$ . It remains to show only that the  $g_j$  and  $g$  agree in some neighborhood of  $\psi_j(D_j \cap S_j)$ . Indeed, when we use the trivializing homeomorphism  $F_j$  to identify  $Y_j$  with  $\mathbb{C} \times (\text{image } \Phi_j)$ , the map  $g_j$  sends  $(z, \beta)$  to  $(a_j z, \beta)$ .  $\square$

## 11. PROOF OF THE LOCAL UNIQUENESS THEOREM.

We now have all the ingredients to prove Theorem 1. We recall the statement:

**Theorem 1.** *Let  $(M, \Phi, \omega, U)$  and  $(M', \Phi', \omega', U)$  be complexity one spaces. Assume that their Duistermaat-Heckman measures are the same, and that their genus and isotropy data over a point  $\alpha \in \mathfrak{t}^*$  are the same. Then there exists a neighborhood of the point  $\alpha$  over which the spaces are isomorphic.*

*Proof.* Since the case that the moment fiber  $\Phi^{-1}(\alpha)$  contains just one orbit is covered by Proposition 2.2, we may assume that the moment fiber contains more than one orbit.

By Lemma 9.7, after possibly restricting to the preimage of a smaller neighborhood of  $\alpha$ , we may assume that  $M$  and  $M'$  are equipped with flattenings. By assumption, the spaces  $M$  and  $M'$  have the same genus and isotropy data. Therefore, by Proposition 10.1, there is a  $\Phi$ -diffeomorphism  $g: M/T \longrightarrow M'/T$ .

Since the spaces have flattenings, Condition (3.2) is satisfied. (See Corollary 9.8.) By assumption, the Duistermaat-Heckman measures of  $M$  and  $M'$  are the same. Hence, we can apply Propositions 3.3 and 4.2. The first implies that the

map  $g$  lifts to a  $\Phi$ - $T$ -diffeomorphism from  $M$  to  $M'$ . The second then guarantees that there exists  $\Phi$ - $T$ -symplectomorphism from  $M$  to  $M'$ .  $\square$

## 12. PROOF OF UNIQUENESS FOR CENTERED SPACES

In this section we prove Theorem 2, which shows that the invariants that we described also separate centered spaces.

We recall Definition 1.5: A proper Hamiltonian  $T$ -manifold  $(M, \omega, \Phi, U)$  is **centered** about a point  $\alpha \in U$  if  $\alpha$  is contained in the closure of the moment image of every orbit type stratum in  $M$ . The local normal form theorem, together with the properness of the moment map, imply that every point in  $\mathfrak{t}^*$  has a neighborhood whose preimage is centered.

We recall the statement of the theorem.

**Theorem 2** (Centered Uniqueness). *Let  $(M, \Phi, \omega, U)$  and  $(M', \Phi', \omega', U)$  be complexity one spaces that are centered about  $\alpha \in U$ . Assume that their Duistermaat-Heckman measures are the same and that their genus and isotropy data over  $\alpha \in \mathfrak{t}^*$  are the same. Then the spaces are isomorphic.*

*Remark 12.1.* In Theorem 2, if the moment fibers  $\Phi^{-1}(\alpha)$  and  $\Phi'^{-1}(\alpha)$  are each a single orbit, the centered spaces are isomorphic if the isotropy data are the same. (The Duistermaat-Heckman measures and the genus are then automatically the same.) The proof for this case only uses three results from earlier sections: Propositions 2.2 and 3.3 and Lemma 5.4.

The following proof of Theorem 2 relies on a couple of technical lemmas which we postpone until after the proof.

*Proof.*

**Case I: the moment fiber is a single orbit.** By Proposition 2.2, there exists a convex sub-neighborhood  $V \subset U$  of  $\alpha$  and a  $\Phi$ - $T$ -diffeomorphism (in fact, symplectomorphism) from  $\Phi^{-1}(V)$  to  $\Phi'^{-1}(V)$ . By Lemma 5.6 we can choose  $V$  so that  $\Phi^{-1}(V)$  and  $\Phi'^{-1}(V)$  satisfy Condition (3.2).

By Lemma 12.2, this implies that there exists a  $\Phi$ - $T$ -diffeomorphism from  $(M, \omega, \Phi)$  to  $(M', \omega', \Phi')$ , and that  $M$  and  $M'$  themselves also satisfy Condition (3.2). The Duistermaat-Heckman measures coincide; hence we may apply Proposition 3.3, which completes the proof.

**Case II: the moment fiber contains more than one orbit.** By Lemma 9.7, there exists a convex sub-neighborhood  $V \subset U$  of  $\alpha$  so that  $\Phi^{-1}(V)$  and  $\Phi'^{-1}(V)$  are equipped with flattenings. By assumption, the spaces  $M$  and  $M'$  have the same genus and isotropy data. Therefore, by Proposition 10.1, there is a  $\Phi$ -diffeomorphism  $g: \Phi^{-1}(V) \rightarrow \Phi'^{-1}(V)$ . Since these spaces have flattenings, Condition (3.2) is satisfied. (See Corollary 9.8.) By assumption, the Duistermaat-Heckman measures of  $M$  and  $M'$  are the same. Hence, Proposition 4.2 implies that the map  $g$  lifts to a  $\Phi$ - $T$ -diffeomorphism from  $\Phi^{-1}(V)$  to  $\Phi'^{-1}(V)$ . Proposition 3.3 completes the proof.  $\square$

We have used the following “stretching lemma”, which tells us that a centered space retracts onto a neighborhood of its central fiber. In a later paper we will use this lemma for spaces which may have complexity greater than one.

**Lemma 12.2.** *Let  $\mathfrak{t}^*$  be the dual of the Lie algebra of a torus  $T$ ,  $U \subset \mathfrak{t}^*$  an open convex neighborhood of a point  $\alpha \in \mathfrak{t}^*$ , and  $V \subset U$  any sub-neighborhood. Then there exists a convex neighborhood  $W$  of  $\alpha$  contained in  $V$ , and a diffeomorphism  $f: U \rightarrow W$  with the following property: for any proper Hamiltonian  $T$ -manifold  $(M, \omega, \Phi, U)$  that is centered about  $\alpha$ , there exists a smooth equivariant orientation preserving diffeomorphism  $F: M \rightarrow \Phi^{-1}(W)$  such that  $\Phi \circ F = f \circ \Phi$ .*

Before beginning the proof of Lemma 12.2, recall that the **Euler vector field** on a vector space  $V$  is given by  $X = \sum x_i \frac{\partial}{\partial x_i}$ , where  $x_i$  are linear coordinates. This vector field is the generator of the flow  $x \mapsto e^t x$ , thus it is independent of the choice of coordinates.

**Lemma 12.3.** *Let  $(M, \omega, \Phi, U)$  be a proper Hamiltonian  $T$ -manifold. Suppose that  $U$  contains the origin  $0$  of  $\mathfrak{t}^*$ , and that the space is centered about the origin. Then the Euler vector field  $X$  on  $\mathfrak{t}^*$  lifts to a smooth invariant vector field  $\tilde{X}$  on  $M$ , that is,  $\Phi_*(\tilde{X}) = X$ .*

*Proof.* By the local normal form theorem, it is enough to construct the vector field  $\tilde{X}$  on the local models. We can then patch together the pieces by an invariant partition of unity.

Notice that if a map  $\Phi: V \rightarrow W$  between vector spaces is homogeneous of degree  $m$ , then  $\Phi_* X_V = m X_W$  where  $X_V$  and  $X_W$  are the Euler vector fields on  $V$  and  $W$ ; this follows from the equality  $\Phi(e^t v) = e^{mt} \Phi(v)$ . In particular, the Euler vector field on  $W$  lifts to a vector field on  $V$ . Similarly, if  $\Phi_i: V_i \rightarrow W_i$ ,  $i = 1, 2$ , are homogeneous (possibly of different degrees), and  $\Phi = \Phi_1 \times \Phi_2: V_1 \times V_2 \rightarrow W_1 \times W_2$ , then the Euler vector field on  $W_1 \times W_2$  lifts to a vector field on  $V_1 \times V_2$ .

Consider a local model in  $M$ , namely,  $Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ , with a moment map  $\Phi_Y([t, z, \nu]) = \alpha + \Phi_H(z) + \nu$ . The stratum fixed by  $H$  is  $T \times_H (\mathbb{C}^n)^H \times \mathfrak{h}^0$ . Because the space is centered, we must have that  $\alpha \in \mathfrak{h}^0$ . Without loss of generality we may assume that  $\alpha = 0$ . To lift the Euler vector field on  $\mathfrak{t}^*$  to a  $T$ -invariant vector field on  $Y$ , it is enough to lift it to an  $H$ -invariant vector field on  $\mathbb{C}^n \times \mathfrak{h}^0$ . This is possible by the previous paragraph, because  $\Phi|_{\mathbb{C}^n \times \mathfrak{h}^0}$  is bihomogeneous.  $\square$

*Proof of Lemma 12.2.* Without loss of generality we may assume that  $\alpha = 0$ . Choose  $\epsilon > 0$  so that an  $\epsilon$ -ball about  $0$  is contained in  $V$ . Let  $g_t: [0, \infty) \rightarrow [0, \infty)$  for  $0 \leq t \leq 1$  be an isotopy such that  $g_0$  is the identity map, the image of  $g_1$  is contained in  $[0, \epsilon)$ ,  $g_t(x) \leq x$  for all  $x$  and all  $t$ , and  $g_t(x) = x$  for all  $x$  near zero and all  $t$ .

Take  $f_t(v) = g_t(|v|) \frac{v}{|v|}$  for all  $v \in U \setminus \{0\}$  and  $f_t(0) = 0$ ; let  $f = f_1$ .

Let  $\xi_t$  be the vector field on  $V$  which generates this isotopy:  $\frac{df_t}{dt} = \xi_t \circ f_t$ . Since  $\xi_t$  vanishes near  $v = 0$ , we can write  $\xi_t = \psi_t \cdot X$ , where  $\psi_t: \mathfrak{t}^* \rightarrow \mathbb{R}$  is a smooth function, and  $X$  is the Euler vector field on  $\mathfrak{t}^*$ . By Lemma 12.3, there exists a smooth invariant vector field  $\tilde{X}$  on  $M$  such that  $\psi_*(\tilde{X}) = X$ . So  $\tilde{\xi}_t = (\psi_t \circ \Phi) \cdot \tilde{X}$  is a smooth invariant vector field on  $M$  which is a lifting of  $\xi_t$ . Because  $\Phi$  is proper, the vector field  $\tilde{\xi}_t$  generates an isotopy,  $F_t$ . Take  $F = F_1$ .  $\square$

### 13. APPLICATION TO PACKINGS OF GRASSMANIANS

We are now ready to present our application. First, we recall a definition from symplectic topology:

**Definition 13.1.** A symplectic manifold  $M$  admits a **full packing by  $k$  equal balls** if for any  $\epsilon > 0$  there exists a symplectic embedding into  $M$  of a disjoint union of  $k$  symplectic balls with equal radii such that the complement of the image has volume less than  $\epsilon$ .

Let  $\text{Gr}^+(2, \mathbb{R}^n)$  denote the Grassmanian of all oriented real 2-planes in  $\mathbb{R}^n$ , together with its canonical (up to multiplication by a constant)  $\text{SO}(n)$ -invariant symplectic structure, and the  $\lfloor \frac{n}{2} \rfloor$  dimensional torus action given by restricting the standard action of  $\text{SO}(n)$ .

**Theorem 3.** *Let  $M$  be the Grassmanian  $\text{Gr}^+(2, \mathbb{R}^5)$  or  $\text{Gr}^+(2, \mathbb{R}^6)$ . There exists an equivariant symplectic embedding of a disjoint union of two symplectic balls with linear actions and with equal radii into  $M$  such that the complement of the image has zero volume. A fortiori, these Grassmanians can be fully packed by two equal balls.*

The following tool will be useful:

**Lemma 13.2.** *Let  $(M, \omega, \Phi)$  be a compact complexity one space over  $\mathfrak{t}^*$ . Let  $p \in M$  be an isolated fixed point with isotropy weights  $\eta_1, \dots, \eta_n$ . Assume that the differences  $\eta_i - \eta_j$  span a codimension one subspace,  $H$ , of  $\mathfrak{t}^*$ . Assume, moreover, that  $p$  is the only fixed point whose moment map image lies on one open side,  $H_+$ , of  $H$ .*

*Then the preimage  $\Phi^{-1}(H_+)$  is equivariantly symplectomorphic to a ball with a linear  $T$ -action.*

*Proof.* First, we show that  $\Phi^{-1}(H_+)$  is centered. The closure  $N$  of an orbit type stratum in  $M$  is itself a compact symplectic manifold with the restricted  $T$  action and moment map. By the convexity theorem, its moment image is the convex hull of the moment images of its fixed points. Either  $N$  contains  $p$ , or its moment image would be contained in  $\text{conv}(M^T \setminus p)$ , and therefore disjoint from  $H_+$ .

Let  $T$  act on  $\mathbb{C}^n$  with weights  $\eta_1, \dots, \eta_n$  and with the moment map that sends  $(z_1, \dots, z_n)$  to  $\Phi(p) + \sum \frac{1}{2} \eta_i |z_i|^2$ . The moment preimage of  $H_+$  in  $\mathbb{C}^n$  is a ball. Hence, this ball is also a centered complexity one space over  $H_+$ .

Since both spaces are centered about  $a = \Phi(p)$ , and the preimages of  $a$  are both single orbits with the same isotropy data, by Theorem 2 the spaces are equivariantly symplectomorphic.  $\square$

Now consider any semi-simple compact Lie group  $G$ , and let  $T$  be a maximal torus. Use the Killing form to identify  $\mathfrak{t}$  and  $\mathfrak{t}^*$  and embed  $\mathfrak{t}^*$  in  $\mathfrak{g}^*$ . Recall that the coadjoint orbit in  $\mathfrak{g}^*$  through an element  $x$  of  $\mathfrak{t}^*$  is a symplectic manifold, and the projection to  $\mathfrak{t}^*$  is a moment map for the  $T$  action. The fixed points for the  $T$ -action are exactly the Weyl group orbit of  $x$  in  $\mathfrak{t}^*$ . Moreover, the isotropy weights at a fixed point  $y \in \mathfrak{t}^*$  are exactly those roots  $\alpha \in \mathfrak{t}^*$  for which  $\langle \alpha, y \rangle < 0$ .

*Proof of Theorem 3 for  $\text{Gr}^+(2, \mathbb{R}^5)$ .* The Lie algebra of the maximal torus of  $\text{SO}(5)$  can be identified with  $\mathbb{R}^2$  with the standard metric. The roots are  $(\pm 1, 0)$ ,  $(0, \pm 1)$ ,  $(\pm 1, \pm 1)$ . The Weyl group acts by permuting the coordinates and by flipping their signs.

The orbit through the point  $(1, 0)$  is naturally identified with the Grassmanian  $\text{Gr}^+(2, \mathbb{R}^5) = \text{SO}(5)/S(\text{O}(2) \times \text{O}(3))$ . The Weyl group orbit of this point consists of the points  $(1, 0)$ ,  $(-1, 0)$ ,  $(0, 1)$ , and  $(0, -1)$ . The image of this moment map is a diamond. The isotropy weights at  $(1, 0)$  are  $(-1, 1)$ ,  $(-1, 0)$ , and  $(-1, -1)$ . By

Lemma 13.2, the preimage of the half space  $\{(x, y) \mid x > 0\}$  is a ball as required. A similar argument shows that the preimage of the opposite half space is again a ball.  $\square$

*Proof of Theorem 3 for  $\text{Gr}^+(2, \mathbb{R}^6)$ .* The Lie algebra of the maximal torus of  $\text{SO}(6)$  can be identified with  $\mathbb{R}^3$  with the standard metric. The Weyl group acts by permuting the coordinates and by flipping the signs of two coordinates at a time. The roots are  $(\pm 1, \pm 1, \pm 1)$ .

The orbit through the point  $(1, 0, 0)$  is naturally identified with the Grassmanian  $\text{Gr}^+(2, \mathbb{R}^6) = \text{SO}(6)/S(O(2) \times O(4))$ . The Weyl group orbit of this point consists of the points  $(\pm 1, 0, 0)$ ,  $(0, \pm 1, 0)$ , and  $(0, 0, \pm 1)$ . The image of this moment map is an octahedron. The isotropy weights at  $(1, 0, 0)$  are  $(-1, \pm 1, \pm 1)$ . By Lemma 13.2, the preimage of the half space  $\{(x, y, z) \mid x > 0\}$  is a ball as required. A similar argument shows that the preimage of the opposite half space is again a ball.  $\square$

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